AUX7

(finally!!!)

Universidad de Chile
Facultad de Ciencias Físicas y Matemáticas
Departamento de Ingeniería Matemática
MA1002-8 Cálculo $\frac{d}{dx}$ e $\int$ · Primavera 2025

Auxiliar 7: Más primitivas

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P1. [Al revés v2]
Calcule las siguientes primitivas:

a) $\displaystyle\int \frac{\cos(x)}{1+\cos(x)}$

b) $\displaystyle\int \frac{1}{3-\sin(x)}$

c) $\displaystyle\int \frac{1}{4+\cos(x)}$

d) $\displaystyle\int \frac{1}{\sin(x)}$

e) $\displaystyle\int \frac{\sin(x)}{1+\sin(x)+\cos(x)}$

f) $\displaystyle\int \frac{x}{(a^2+x^2)^{\frac{3}{2}}}\,dx$

g) $\displaystyle\int \frac{x}{(a^2+x^2)^{\frac{1}{2}}}\,dx$

h) $\displaystyle\int \frac{1}{(a^2+x^2)^{\frac{3}{2}}}\,dx$

P2. [Más recurrencias v2]
Sea $I_n = \displaystyle\int \frac{\cos(nx)}{\cos^n(x)}\,dx$ definido para $n \in \mathbb{N}$. El objetivo de este problema es encontrar una ecuación de recurrencia para expresar I_{n+1} en términos de I_n.

a) Calcule I_1 e I_2.

b) Compute $\displaystyle\int \frac{\sin(x)}{\cos^{n+1}(x)}\,dx$

c) Determine la expresión buscada.

P3. [Despejando]
Considere $I := \displaystyle\int \cos(\ln(x))\,dx$ y $J := \displaystyle\int \sin(\ln(x))\,dx$.

a) Plantee un sistema lineal que permita calcular I y J. INDICACIÓN: Use integración por partes.

b) Calcule I y J.

P4. [Pincelada]
Usando la partición $P = \left\{0, \frac{1}{2}, 1\right\}$, demuestre que:

$$\frac{1}{2}\left(\frac{1}{e^{\frac{1}{4}}}+\frac{1}{e}\right) \leq \int_0^1 e^{-x^2}\,dx \leq \frac{1}{2}\left(1+\frac{1}{e^{\frac{1}{4}}}\right)$$

P_1 a)

a) $\displaystyle\int \frac{\cos(x)}{1+\cos(x)}\,dx =: I$

$\displaystyle\int$ fración de $\cos(\cdot),\operatorname{sen}(\cdot)$ $\rightsquigarrow$ nueva variable $= \operatorname{tg}\left(\frac{1}{2}\text{ variable actual}\right)$

Usando c.v. $\displaystyle u=\operatorname{tg}\left(\tfrac{1}{2}x\right) \Rightarrow dx=\frac{2}{1+u^2}du \;;\; \cos(x)=\frac{1-u^2}{1+u^2}$

$$\Rightarrow I=\int \frac{\left(\frac{1-u^2}{1+u^2}\right)}{1+\left(\frac{1-u^2}{1+u^2}\right)}\cdot\frac{2}{1+u^2}du = 2\int\frac{\frac{1-u^2}{1+u^2}}{\frac{1+\cancel{u^2}+1\cancel{-u^2}}{1+u^2}}\cdot\frac{1}{1+u^2}du$$

$$=2\int\frac{\frac{1-u^2}{1+u^2}}{\frac{2}{1+u^2}}\cdot\frac{1}{1+u^2}du = \cancel{2}\int\frac{1-u^2}{\cancel{1+u^2}}\cdot\frac{\cancel{1+u^2}}{\cancel{2}}\cdot\frac{1}{1+u^2}du=\int\frac{1-u^2}{1+u^2}du$$

quisiera encontrar en términos del numerado un factor del denominador (para simplificar la integración)

$$=\int\frac{-(1+u^2)+2}{1+u^2}du=-\int\frac{1+u^2}{1+u^2}du+2\int\frac{1}{1+u^2}du=-\int du+2\int\frac{1}{1+u^2}du$$

$$=-u+2\operatorname{arctg}(u)+\mu,\ \mu\in\mathbb{R} = -\operatorname{tg}\left(\tfrac{1}{2}x\right)+2\cdot\tfrac{1}{2}x+\mu,\ \mu\in\mathbb{R} = -\operatorname{tg}\left(\tfrac{1}{2}x\right)+x+\mu,\ \mu\in\mathbb{R}$$

CAMBIO DE VARIABLE TRIGONOMÉTRICO PARA EXPRESIONES RACIONALES QUE CONTENGAN, sen(·), cos(·)

$$\int \frac{f(\operatorname{sen}(\theta),\cos(\theta))}{g(\operatorname{sen}(\theta),\cos(\theta))}\,d\theta \rightsquigarrow \boxed{\underset{\text{nueva variable}}{t} = \operatorname{tg}\big(\tfrac{1}{2}\underset{\text{variable actual}}{\theta}\big)}$$

Permite expresar sen(·), cos(·) con polinomios (fracciones)

para deducir → expresar sen(·), cos(·) en términos de t

$$\boxed{d\theta = \frac{2}{1+t^2}\,dt}$$

$$\boxed{\cos(\theta) = \frac{1-t^2}{1+t^2}}$$

$$\boxed{\operatorname{sen}(\theta) = \frac{2t}{1+t^2}}$$

$t = \operatorname{tg}\left(\frac{1}{2}\theta\right)$

$\Rightarrow \operatorname{arctg}(t) = \frac{1}{2}\theta$

$\Rightarrow 2\operatorname{arctg}(t) = \theta$

$\Rightarrow d\theta = 2\,\frac{1}{1+t^2}\,dt$

$$\boxed{\cos(2\theta) = 2\cos^2(\theta) - 1}$$

$$\Rightarrow \cos(\theta) = \boxed{2\cos^2\left(\tfrac{1}{2}\theta\right) - 1}$$

$$= 2\cdot\frac{1}{\sec^2\left(\frac{1}{2}\theta\right)} - 1 \;;$$

$$\frac{\operatorname{sen}^2(\cdot)}{\cos^2(\cdot)} + \frac{\cos^2(\cdot)}{\cos^2(\cdot)} = \frac{1}{\cos^2(\cdot)}$$

$$\Leftrightarrow \operatorname{tg}^2(\cdot) + 1 = \sec^2(\cdot)$$

$$= 2\cdot\frac{1}{\left[1+\operatorname{tg}^2\left(\frac{1}{2}\theta\right)\right]} - 1$$

$$= 2\,\frac{1}{1+t^2} - 1$$

$$= \frac{2-1-t^2}{1+t^2} = \boxed{\frac{1-t^2}{1+t^2}}$$

$$\cos^2\left(\tfrac{1}{2}\theta\right) = \left(\frac{1-t^2}{1+t^2} + 1\right)\frac{1}{2}$$

$$= \left(\frac{1\cancel{-t^2}+1\cancel{+t^2}}{1+t^2}\right)\frac{1}{2} = \frac{1}{1+t^2}$$

$$\Rightarrow \underbrace{\sqrt{\cos^2\left(\tfrac{1}{2}\theta\right)}}_{=\cos\left(\frac{1}{2}\theta\right)} = \sqrt{\frac{1}{1+t^2}}$$

$$\boxed{\operatorname{sen}(2\theta) = 2\operatorname{sen}(\theta)\cos(\theta)}$$

$$\Rightarrow \operatorname{sen}(\theta) = 2\,\underbrace{\operatorname{sen}\left(\tfrac{1}{2}\theta\right)}\,\underbrace{\cos\left(\tfrac{1}{2}\theta\right)}$$

$$= 2\sqrt{\frac{t^2}{1+t^2}}\,\sqrt{\frac{1}{1+t^2}}$$

$$= 2\,\frac{t}{1+t^2}$$

$$\cos^2\left(\tfrac{1}{2}\theta\right) + \operatorname{sen}^2\left(\tfrac{1}{2}\theta\right) = 1$$

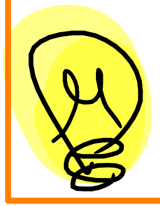
$$\Leftrightarrow \frac{1}{1+t^2} + \operatorname{sen}^2\left(\tfrac{1}{2}\theta\right) = 1$$

$$\Leftrightarrow \operatorname{sen}^2\left(\tfrac{1}{2}\theta\right) = \frac{t^2}{1+t^2}$$

$$\Rightarrow \underbrace{\sqrt{\operatorname{sen}^2\left(\tfrac{1}{2}\theta\right)}}_{\operatorname{sen}\left(\frac{1}{2}\theta\right)} = \sqrt{\frac{t^2}{1+t^2}}$$

P_1 b)

b) $\int \frac{1}{3-5\sin(x)}\,dx = I$



$\int$ fración de $\cos(\cdot), \operatorname{sen}(\cdot)$ $\rightsquigarrow$ nueva variable $= \operatorname{tg}\left(\frac{1}{2}\text{ variable actual}\right)$

Cambio de variable: $\Rightarrow dx = \frac{2\,du}{1+u^2}$ y $\operatorname{sen}(x) = \frac{2u}{1+u^2}$

$$\Rightarrow I = \int \frac{1}{3-5\left(\frac{2u}{1+u^2}\right)} \frac{2}{1+u^2}\,du = \int \frac{1}{\frac{3(1+u^2)-10u}{1+u^2}}\,\frac{2}{1+u^2}\,du = \int \frac{\cancel{1+u^2}}{3u^2-10u+3}\,\frac{2}{\cancel{1+u^2}}\,du$$

$$= 2\int \frac{1}{3u^2-10u+3}\,du = 2\int \frac{1}{(u-3)(3u-1)}\,du$$

$1 \quad -3$

$3 \quad -1$

Por F.P.'s: $\frac{1}{(u-3)(3u-1)} \overset{!}{=} \frac{A}{u-3} + \frac{B}{3u-1}$ $\quad /\cdot(u-3)(3u-1)$

$\Rightarrow 1 = A(3u-1) + B(u-3)$

$\Rightarrow 1 = (3A+B)u + (-A-3B)$

$\Leftrightarrow -A-3B = 1 \wedge 3A+B = 0$

$$\begin{cases} -A - 3B = 1 & (1) \\ 3A + B = 0 & (2) \end{cases}$$

$(1) + 3\cdot(2)$: $\quad 8A = 1$

$$\boxed{A = \frac{1}{8}} \quad (A)$$

$(A) \to (2)$: $\quad 3\cdot\frac{1}{8} + B = 0 \Rightarrow \boxed{B = -\frac{3}{8}}$

$$\Rightarrow I = 2\int \left[\frac{1}{8}\cdot\frac{1}{u-3} - \frac{3}{8}\,\frac{1}{3u-1}\right] du$$

$$= 2\cdot\frac{1}{8}\int\frac{1}{u-3}\,du - 2\cdot\frac{3}{8}\int\frac{1}{3u-1}\,du$$

$$\boxed{[\ln(f)]' = \frac{1}{f}f'} \qquad f>0$$

$$= \frac{1}{4}\int\frac{1}{u-3}\,du - \cancel{\frac{3}{4}}\cdot\cancel{\frac{1}{3}}\int\frac{3}{3u-1}\,du = \frac{1}{4}\ln(|u-3|) - \frac{1}{4}\ln(|3u-1|) + \mu,\ \mu\in\mathbb{R}$$

$$\Rightarrow I = \frac{1}{4}\left[\ln(|u-3|) - \ln(|3u-1|)\right] + \mu, \mu \in \mathbb{R}$$

$$= \frac{1}{4}\ln\left(\left|\frac{u-3}{3u-1}\right|\right) + \mu, \mu \in \mathbb{R}$$

$$\boxed{u = tg\left(\tfrac{1}{2}x\right)}$$

$$= \frac{1}{4}\ln\left(\left|\frac{tg\left(\frac{1}{2}x\right)-3}{3tg\left(\frac{1}{2}x\right)-1}\right|\right) + \mu, \mu \in \mathbb{R}$$

$\boxtimes$

P_1 e)

c) $\int \frac{1}{4+3\cos(x)}\,dx = I$

$\int$ fración de $\cos(\cdot), \operatorname{sen}(\cdot)$ ⟶ nueva variable $= \operatorname{tg}\left(\frac{1}{2}\text{ variable actual}\right)$

Sea el C.V. $t = \operatorname{tg}\left(\frac{1}{2}x\right) \Longrightarrow \cos(x) = \frac{1-t^2}{1+t^2}$ y $dx = \frac{2}{1+t^2}dt$

$$\Rightarrow I = \int \frac{1}{4+3\left(\frac{1-t^2}{1+t^2}\right)}\,\frac{2}{1+t^2}\,dt = \int \frac{1}{\frac{4(1+t^2)+3(1-t^2)}{1+t^2}}\,\frac{2}{1+t^2}\,dt = \int \frac{\cancel{1+t^2}}{7+t^2}\,\frac{2}{\cancel{1+t^2}}\,dt$$

$$= 2\int \frac{1}{7+t^2}\,dt$$

$\iff \frac{1}{\sqrt{7}}t = \operatorname{tg}(w) \iff \boxed{\operatorname{arctg}\left(\frac{1}{\sqrt{7}}t\right) = w}$

$t = \sqrt{7}\operatorname{tg}(w) \Rightarrow t^2 = 7\operatorname{tg}^2(w) \Rightarrow 7+t^2 = 7+7\operatorname{tg}^2(w) \iff \boxed{7+t^2 = 7\sec^2(w)}$

$\frac{\operatorname{sen}^2(\cdot)}{\cos^2(\cdot)} + \frac{\cos^2(\cdot)}{\cos^2(\cdot)} = \frac{1}{\cos^2(\cdot)}$

$\iff \operatorname{tg}^2(\cdot) + 1 = \sec^2(\cdot)$

$\iff 7\operatorname{tg}^2(\cdot) + 7 = 7\sec^2(\cdot)$

$\Rightarrow \boxed{dt = \sqrt{7}\sec^2(w)\,dw}$

$$= 2\int \frac{1}{7\cancel{\sec^2(w)}}\sqrt{7}\cancel{\sec^2(w)}\,dw = \frac{2}{7}\sqrt{7}\int dw = \frac{2}{7}\sqrt{7}\,w + \mu,\ \mu \in \mathbb{R};$$

⚠ $\boxed{w = \operatorname{arctg}\left(\frac{1}{\sqrt{7}}t\right)}$

$$\Rightarrow I = \frac{2}{7}\sqrt{7}\operatorname{arctg}\left(\frac{1}{\sqrt{7}}t\right) + \mu,\ \mu \in \mathbb{R}$$

$$\Rightarrow I = \frac{2}{\sqrt{7}} \operatorname{arctg}\left(\frac{1}{\sqrt{7}} t\right) + \mu, \mu \in \mathbb{R}; \quad ⚠ \quad t = \operatorname{tg}\left(\tfrac{1}{2}x\right)$$

$$\frac{\sqrt{7}}{7} = \frac{1}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}}$$

$$\Rightarrow I = \frac{2}{\sqrt{7}} \operatorname{arctg}\left(\frac{1}{\sqrt{7}} \operatorname{tg}\left(\tfrac{1}{2}x\right)\right) + \mu, \mu \in \mathbb{R}. \quad \square$$

$(P_1 d)$

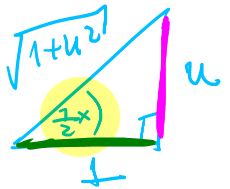
d) $\int \frac{1}{\sin(x)}\,dx =: I$

$\int$ fración de $\cos(\cdot), sen(\cdot)$ $\rightsquigarrow$ nueva variable $= tg\left(\frac{1}{2}\text{ variable actual}\right)$

cateto opuesto

Sea el c.V. $\frac{u}{1} = tg\left(\frac{1}{2}x\right) \Rightarrow dx = \frac{2}{1+u^2}du$ y $sen(x) = \frac{2u}{1+u^2}$

cateto adyacente



$\Rightarrow sen\left(\frac{1}{2}x\right) = \frac{u}{\sqrt{1+u^2}}$

$\Rightarrow \cos\left(\frac{1}{2}x\right) = \frac{1}{\sqrt{1+u^2}}$

$\Rightarrow sen(x) = 2 sen\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)$

$= 2\,\frac{u}{1+u^2}$

$(sen(2x) = 2 sen(x)\cos(x))$

$$\Rightarrow I = \int \frac{1}{\frac{\cancel{2}u}{\cancel{1+u^2}}} \frac{\cancel{2}}{\cancel{1+u^2}}\,du = \int \frac{1}{u}\,du = \ln(|u|) + \mu,\ \mu \in \mathbb{R}$$

$u = u(x)$

$$= \ln\left(\left|tg\left(\tfrac{1}{2}x\right)\right|\right) + \mu,\ \mu \in \mathbb{R}$$

$(P_1\, e)$

e) $\int \frac{\sin(x)}{1+\sin(x)+\cos(x)}\,dx =: I$

$\int$ fración de $\cos(\cdot), sen(\cdot)$ $\rightsquigarrow$ nueva variable $= tg\left(\frac{1}{2}\text{ variable actual}\right)$

Sea el c.V. $w = tg\left(\frac{1}{2}x\right) \Rightarrow \boxed{dx = \frac{2}{1+w^2}\,dw}$ y $\boxed{\cos(x) = \frac{1-w^2}{1+w^2}}$ y $\boxed{sen(x) = \frac{2w}{1+w^2}}$

$$\Rightarrow I = \int \frac{\frac{2w}{1+w^2}}{1+\frac{2w}{1+w^2}+\frac{1-w^2}{1+w^2}}\,\frac{2}{1+w^2}\,dw$$

$$= \int \frac{\frac{2w}{1+w^2}}{\frac{1(1+\cancel{w^2})+2w+1-\cancel{w^2}}{1+w^2}}\,\frac{2}{1+w^2}\,dw = \int \frac{\frac{2w}{1+w^2}}{\frac{2+2w}{1+w^2}}\,\frac{2}{1+w^2}\,dw$$

$$= \int \frac{\cancel{2}w}{\cancel{1+w^2}}\,\frac{\cancel{1+w^2}}{\cancel{2}(1+w)}\,\frac{2}{1+w^2}\,dw = 2\int \frac{w}{(1+w)(1+w^2)}\,dw$$

Usando F.P.

$$\frac{w}{(1+w)(1+w^2)} \overset{!}{=} \frac{A}{1+w} + \frac{Bw+C}{1+w^2} \quad / \cdot (1+w)(1+w^2)$$

$$\Rightarrow w = A(1+w^2) + (Bw+C)(1+w)$$

$$\Leftrightarrow w = A + Aw^2 + Bw + C + Bw^2 + Cw$$

$$\Leftrightarrow w = (A+B)w^2 + (B+C)w + (A+C)$$

$$\Leftrightarrow \begin{cases} A+B=0 & (1) \\ B+C=1 & (2) \\ A+C=0 & (3) \end{cases}$$

$(1)-(3)$: $B-C=0$

$\Leftrightarrow B=C$ (4)

$(4) \rightarrow (2)$: $B+B=1$

$\Leftrightarrow 2B=1$

$\Leftrightarrow \boxed{B=\frac{1}{2}=C}$ (Δ)

$(\Delta) \rightarrow (3)$: $\boxed{A=-\frac{1}{2}}$

$$\Rightarrow I = 2\int\left[-\frac{1}{2}\frac{1}{1+w} + \frac{1}{2}\frac{w}{1+w^2} + \frac{1}{2}\frac{1}{1+w^2}\right]dw$$

$$= 2\left(-\frac{1}{2}\right)\int\frac{1}{1+w}dw + 2\frac{1}{2}\int\frac{w}{1+w^2}dw + 2\frac{1}{2}\int\frac{1}{1+w^2}dw$$

$$= -\int\frac{1}{1+w}dw + \frac{1}{2}\int\frac{2w}{1+w^2}dw + \int\frac{1}{1+w^2}dw$$

$$[\ln(f)]' = \frac{1}{f}f' \quad (f>0)$$

$$= -\ln(|1+w|) + \frac{1}{2}\ln(|1+w^2|) + \operatorname{arctg}(w) + \mu, \ \mu\in\mathbb{R}$$

$$= \ln\left(\frac{\sqrt{1+w^2}}{|1+w|}\right) + \operatorname{arctg}(w) + \mu, \ \mu\in\mathbb{R}$$

$$w = \operatorname{tg}\left(\tfrac{1}{2}x\right)$$

$$= \ln\left(\frac{\sqrt{1+\operatorname{tg}^2\left(\frac{1}{2}x\right)}}{\left|1+\operatorname{tg}\left(\frac{1}{2}x\right)\right|}\right) + \operatorname{arctg}\left(\operatorname{tg}\left(\tfrac{1}{2}x\right)\right) + \mu, \ \mu\in\mathbb{R}$$

$$\frac{\operatorname{sen}^2(\cdot)}{\cos^2(\cdot)} + \frac{\cos^2(\cdot)}{\cos^2(\cdot)} = \frac{1}{\cos^2(\cdot)}$$

$$\Longleftrightarrow \operatorname{tg}^2(\cdot) + 1 = \sec^2(\cdot)$$

$$\Longrightarrow I = \ln\left(\frac{\sqrt{\sec^2\left(\frac{1}{2}x\right)}}{\left|1+\operatorname{tg}\left(\frac{1}{2}x\right)\right|}\right) + \frac{1}{2}x + \mu, \quad \mu \in \mathbb{R}$$

$$= \ln\left(\left|\frac{\sec\left(\frac{1}{2}x\right)}{1+\operatorname{tg}\left(\frac{1}{2}x\right)}\right|\right) + \frac{1}{2}x + \mu, \quad \mu \in \mathbb{R} \quad \blacksquare$$

Vimos que numerador es derivada de algo del denominador
→ conviene cambio de variable

P1 f) g) h)

Si lo anterior no ocurre, y el denominador es de la forma constante + variable²
→ conviene C.V. $\left(\begin{array}{c}\text{variable}\\ \text{actual}\end{array}\right) = \sqrt{\text{constante}}\ \operatorname{tg}\left(\begin{array}{c}\text{variable}\\ \text{nueva}\end{array}\right)$

f) $\int \frac{x}{(a^2+x^2)^{\frac{3}{2}}}\,dx =: I$

Sea el c.v. $u = a^2 + x^2$

$\Rightarrow du = 2x\,dx$

$\Leftrightarrow \frac{1}{2}du = x\,dx$

$$\Rightarrow I = \int \frac{1}{u^{\frac{3}{2}}}\,\frac{1}{2}du$$

$$= \frac{1}{2}\int u^{-\frac{3}{2}}du$$

$$= \frac{1}{2}\,\frac{1}{-\frac{3}{2}+1}\,u^{-\frac{3}{2}+1} + \mu,\ \mu\in\mathbb{R}$$

$$= \frac{1}{\cancel{2}}(-\cancel{2})\,u^{-\frac{1}{2}} + \mu,\ \mu\in\mathbb{R}$$

⚠ devolverse a la var. original

$$= -(a^2+x^2)^{-\frac{1}{2}} + \mu,\ \mu\in\mathbb{R}$$ ◪

g) $\int \frac{x}{(a^2+x^2)^{\frac{1}{2}}}\,dx =: I$

Sea el c.v. $u = a^2 + x^2$

$\Rightarrow du = 2x\,dx$

$\Leftrightarrow \frac{1}{2}du = x\,dx$

$$\Rightarrow I = \int \frac{1}{u^{\frac{1}{2}}}\,\frac{1}{2}du$$

$$= \frac{1}{2}\int u^{-\frac{1}{2}}du$$

$$= \frac{1}{2}\,\frac{1}{-\frac{1}{2}+1}\,u^{-\frac{1}{2}+1} + \mu,\ \mu\in\mathbb{R}$$

$$= \frac{1}{2}\cdot 2\,u^{\frac{1}{2}} + \mu,\ \mu\in\mathbb{R}$$

$$= u^{\frac{1}{2}} + \mu,\ \mu\in\mathbb{R}$$ ⚠ $u = a^2 + x^2$

$$\Rightarrow I = (a^2+x^2)^{\frac{1}{2}} + \mu,\ \mu\in\mathbb{R}$$ ◪

$a > 0$

$\Rightarrow \sqrt{a^2} = |a| = a$

h) $\int \frac{1}{(a^2+x^2)^{\frac{3}{2}}}\,dx =: I$

$t = \arctan\left(\frac{x}{a}\right)$

Sea el c.v. $x = \sqrt{a^2}\,tg(t)$

$\frac{sen^2(\cdot)}{cos^2(\cdot)} + \frac{cos^2(\cdot)}{cos^2(\cdot)} = \frac{1}{cos^2(\cdot)}$

$\Leftrightarrow tg^2(\cdot) + 1 = sec^2(\cdot)$

$\Rightarrow x^2 = a^2 tg^2(t)$

$\Rightarrow a^2 + x^2 = a^2 + a^2 tg^2(t) = a^2 sec^2(t)$

$\Rightarrow (a^2+x^2)^{\frac{3}{2}} = (a^2 sec^2(t))^{\frac{3}{2}} = a^3 sec^3(t)$

$* \Rightarrow dx = a\,sec^2(t)\,dt$

$$\Rightarrow I = \int \frac{1}{\cancel{a^3}\,\cancel{sec^3}(t)}\,\cancel{a}\,\cancel{sec^2}(t)\,dt$$

$$= \frac{1}{a^2}\int \frac{1}{sec(t)}\,dt = \frac{1}{a^2}\int cos(t)\,dt$$

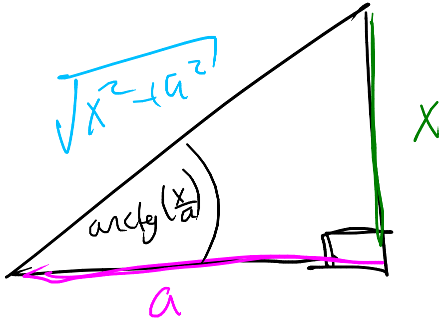
$$= \frac{1}{a^2}\,sen(t) + \mu,\ \mu\in\mathbb{R}$$

$$= \frac{1}{a^2}\,sen\left(\arctan\left(\frac{x}{a}\right)\right) + \mu,\ \mu\in\mathbb{R}$$

cateto opuesto

$$\beta := \operatorname{arctg}\left(\frac{x}{a}\right)$$

cateto adyacente



$$\operatorname{sen}\left(\operatorname{arctg}\left(\frac{x}{a}\right)\right) = \frac{x}{\sqrt{x^2+a^2}}$$

$$\Rightarrow I = \frac{1}{a^2}\frac{x}{\sqrt{x^2+a^2}} + \mu, \quad \mu \in \mathbb{R}$$

P2

P2. [Más recurrencias v2]

Sea $I_n = \int \frac{\cos(nx)}{\cos^n(x)}\,dx$ definido para $n \in \mathbb{N}$. El objetivo de este problema es encontrar una ecuación de recurrencia para expresar I_{n+1} en términos de I_n.

a) Calcule I_1 e I_2.

b) Compute $\int \frac{\sin(x)}{\cos^{n+1}(x)}\,dx$

c) Determine la expresión buscada.

P2 a)

$$* I_1 := \int \frac{\cos(1\cdot x)}{\cos^1(x)}\,dx = \int \frac{\cos(x)}{\cos(x)}\,dx = \int 1\,dx = x + \mu,\ \mu \in \mathbb{R} //$$

$$* I_2 := \int \frac{\cos(2x)}{\cos^2(x)}\,dx = \int \frac{\cos^2(x)}{\cos^2(x)}\,dx - \int \frac{\operatorname{sen}^2(x)}{\cos^2(x)}\,dx = \int 1\,dx - \int tg^2(x)\,dx$$

$\cos(2x) = \cos^2(x) - \operatorname{sen}^2(x)$

$\frac{\cos^2(\cdot)+\operatorname{sen}^2(\cdot)}{\cos^2(\cdot)} = \frac{1}{\cos^2(\cdot)}$

$\Leftrightarrow 1 + tg^2(\cdot) = \sec^2(\cdot)$

$$= x - \int (\sec^2(x) - 1)\,dx$$

$$= x - \int \sec^2(x)\,dx + \int dx$$

$$= x - tg(x) + x + \mu,\ \mu \in \mathbb{R}$$

$$= 2x - tg(x) + \mu,\ \mu \in \mathbb{R}$$

Calcularlo por def. !!

P2 b) $\displaystyle\int \frac{\operatorname{sen}(x)}{\cos^{n+1}(x)}\,dx =: I$

buscar si es de la forma $\displaystyle\int \frac{f'}{f}$

$u = \cos(x) \Rightarrow du = -\operatorname{sen}(x)\,dx$

$\Leftrightarrow -du = \operatorname{sen}(x)\,dx$

$$\Rightarrow I = \int \frac{1}{u^{n+1}}(-du) = -\int \frac{1}{u^{n+1}}\,du = -\int u^{-(n+1)}\,du = -\frac{1}{[-(n+1)]+1}\,u^{[-(n+1)]+1} + \mu,\ \mu \in \mathbb{R}$$

$$= -\frac{1}{-n-1+1}\,u^{-n} + \mu,\ \mu \in \mathbb{R}$$

$$= \frac{1}{n}u^{-n} + \mu,\ \mu \in \mathbb{R}$$

$$= \frac{1}{n}\frac{1}{u^{n}} + \mu,\ \mu \in \mathbb{R}$$

⚠ $u = u(x) = \cos(x)$

$$\Rightarrow I = \frac{1}{n}\frac{1}{\cos(x)^{n}} + \mu,\ \mu \in \mathbb{R} \quad \blacksquare$$

(P2) c) Vamos a usar I_{n+1} por def. / y la calculamos

$$\Rightarrow I_{n+1} = \int \frac{\cos((n+1)x)}{\cos^{n+1}(x)}\,dx \qquad ; \cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \operatorname{sen}(\alpha)\operatorname{sen}(\beta)$$

$$= \int \frac{\cos(nx+x)}{\cos^{n+1}(x)}\,dx$$

$$= \int \frac{\cos(nx)\cos(x) - \operatorname{sen}(nx)\operatorname{sen}(x)}{\cos^{n+1}(x)}\,dx$$

$$= \int \frac{\cos(nx)\cancel{\cos(x)}}{\cos^{n\cancel{+1}}(x)}\,dx - \int \frac{\operatorname{sen}(nx)\operatorname{sen}(x)}{\cos^{n+1}(x)}\,dx$$

$$= \underbrace{\int \frac{\cos(nx)}{\cos^n(x)}\,dx}_{I_n} - \underbrace{\int \frac{\operatorname{sen}(nx)\,\operatorname{sen}(x)}{\cos^{n+1}(x)}\,dx}_{i.p.p.}$$

porque en (P2) b) calculamos su integral

i.p.p. $\begin{cases} u = \operatorname{sen}(nx) \longrightarrow du = n\cos(nx)\,dx & \text{“+ fácil de derivar”} \\ dv = \dfrac{\operatorname{sen}(x)}{\cos^{n+1}(x)}\,dx \rightarrow v = \dfrac{1}{n}\,\dfrac{1}{\cos^n(x)} & \text{“+ fácil de integrar”} \end{cases}$

$$= uv - \int v\,du$$

$$= \frac{1}{n}\left[\operatorname{sen}(nx)\,\frac{1}{\cos^n(x)}\right] - \frac{1}{\cancel{n}}\underbrace{\int \frac{\cancel{n}\cos(nx)}{\cos^n(x)}\,dx}_{I_n}$$

$$\Rightarrow I_{n+1} = I_n - \left(\frac{1}{n}\,\frac{\operatorname{sen}(nx)}{\cos^n(x)} - I_n\right) = I_n - \frac{1}{n}\,\frac{\operatorname{sen}(nx)}{\cos^n(x)} + I_n$$

$$\Rightarrow \boxed{I_{n+1} = 2I_n - \frac{1}{n}\,\frac{\operatorname{sen}(nx)}{n\cos^n(x)}}$$

P3

P3. [Despejando]

Considere $I := \int \cos(\ln(x))\, dx$ y $J := \int \sin(\ln(x))\, dx$.

a) Plantee un sistema lineal que permita calcular I y J. INDICACIÓN: Use integración por partes

b) Calcule I y J.

P3 a)

$\boxed{\text{Integrar por partes, para "rescatar" una expresión de la otra}}$

$$* \; I = \int \cos(\ln(x))\, dx = \int 1 \cdot \cos(\ln(x))\, dx$$

i. p. p.

$$u = \cos(\ln(x)) \Rightarrow du = -\operatorname{sen}(\ln(x)) [\ln(x)]' dx = -\operatorname{sen}(\ln(x)) \frac{1}{x} dx$$

$$dv = 1 dx \Rightarrow v = x$$

$$\Rightarrow I = uv - \int v\, du = x \cos(\ln(x)) + \int \frac{\operatorname{sen}(\ln(x))}{\cancel{x}} \cancel{x}\, dx$$

$$\Rightarrow I = x\cos(\ln(x)) + \underbrace{\int \operatorname{sen}(\ln(x))\,dx}_{J}$$

$$\Rightarrow \boxed{I = x\cos(\ln(x)) + J}$$

$$* \quad J = \int \operatorname{sen}(\ln(x))\,dx = \int 1\cdot \operatorname{sen}(\ln(x))\,dx$$

i.p.p. $\begin{cases} u = \operatorname{sen}(\ln(x)) \Rightarrow du = \cos(\ln(x))\,[\ln(x)]'\,dx = \cos(\ln(x))\,\frac{1}{x}\,dx \\ dv = 1\,dx \Rightarrow v = x \end{cases}$

$$\Rightarrow J = uv - \int v\,du = x\operatorname{sen}(\ln(x)) - \int \cos(\ln(x))\,\frac{1}{\cancel{x}}\,\cancel{x}\,dx$$

$$\Rightarrow J = x\operatorname{sen}(\ln(x)) - \int \cos(\ln(x))\,dx \quad I \quad \Rightarrow \boxed{J = x\operatorname{sen}(\ln(x)) - I}$$

$$\begin{cases} I = x\cos(\ln(x)) + J \\ J = x\,\mathrm{sen}(\ln(x)) - I \end{cases}$$

P_3 b) $\begin{cases} I - J = x\cos(\ln(x)) & (1) \\ I + J = x\,\mathrm{sen}(\ln(x)) & (2) \end{cases}$

Resolver sistema con herramientas "simples"

reducción

(1) + (2): $2I = x\left(\cos(\ln(x)) + \mathrm{sen}(\ln(x))\right)$

$$\Rightarrow \boxed{I = \frac{1}{2}x\left(\cos(\ln(x)) + \mathrm{sen}(\ln(x))\right)}$$

(2) − (1): $2J = x\left(\mathrm{sen}(\ln(x)) - \cos(\ln(x))\right) \Rightarrow \boxed{J = \frac{1}{2}x\left(\mathrm{sen}(\ln(x)) - \cos(\ln(x))\right)}$

P4

P4. [Pincelada]

Usando la partición $P = \left\{0, \frac{1}{2}, 1\right\}$, demuestre que:

$$\frac{1}{2}\left(\frac{1}{e^{\frac{1}{4}}} + \frac{1}{e}\right) \leq \int_0^1 e^{-x^2}\, dx \leq \frac{1}{2}\left(1 + \frac{1}{e^{\frac{1}{4}}}\right)$$

Quiero acotar por arriba y por abajo integral (fn continua) → Sumas de Riemann

Para definir sumas de Riemann, requiero → partición → equiespaciada
→ función

Para definir sumas de Rieman, requerimos $M(f)$, $m(f)$ en Δx_i, $i \in [1..n]$

↳ requerimos ver si la función es creciente o decreciente

Calculamos las Sumas de Riemann para $\left\{ f : [0,1] \to \mathbb{R},\ x \mapsto f(x) = e^{x} \right.$ → continua → RI (y acotada)

Requerimos una partición → equiespaciada

$$\to x_i = a + \frac{i}{n}(b-a) \quad ; \quad [0,1] =: [a,b]$$

$$\Rightarrow x_i = 0 + \frac{i}{n}(1) \Leftrightarrow \boxed{x_i = \frac{i}{n}}$$

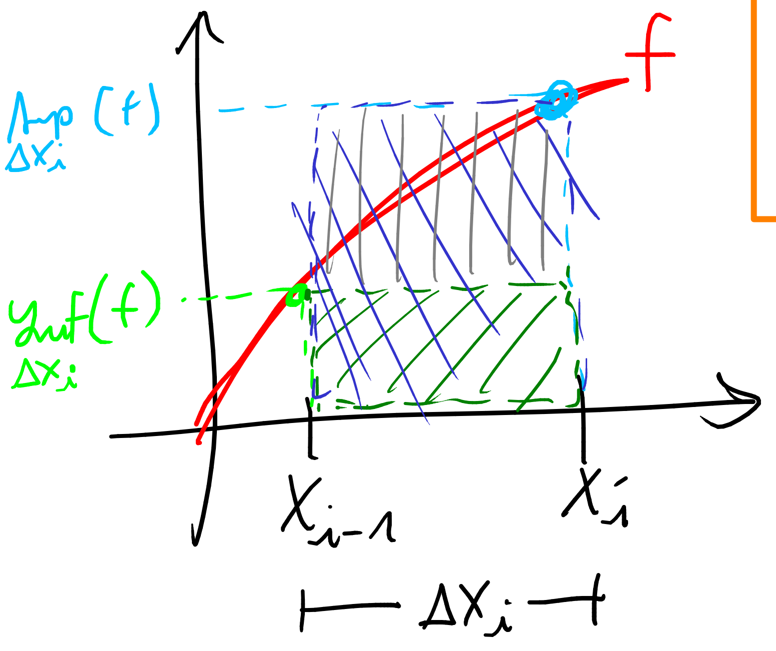
$$\Rightarrow \boxed{\Delta x_i = \frac{1}{n}} \qquad \boxed{\Delta x_i = x_i - x_{i-1}}$$

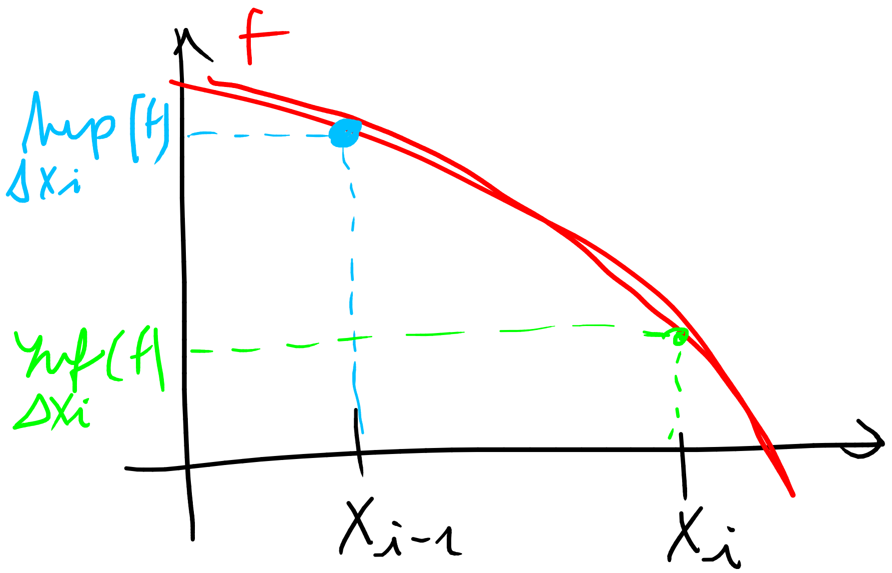
Que coincide con $P = \{0, \frac{1}{2}, 1\}$ partición equiespaciada de $n=3$

Notar que:

$$M(f) := \sup_{x\in[x_{i-1},x_i]} (f(x))$$

$$m(f) := \inf_{x\in[x_{i-1},x_i]} (f(x))$$





f creciente

$M(f) = f(x_i)$

$m(f) = f(x_{i-1})$

$e^{-x^2} \to$ f decreciente

$M(f) = f(x_{i-1})$

$m(f) = f(x_i)$

Como $f(x) = e^{-x^2}$ es decreciente: $\begin{cases} M(f) = f(x_{i-1}) \\ m(f) = f(x_i) \end{cases}$.

Usando que: $\left\{ \boxed{\Delta x_i = \frac{1}{n}} \quad \boxed{x_i = \frac{i}{n}} \quad \boxed{n=2} \quad \boxed{[0,1] =: [a,b]} \right.$

y que por def. $\left\{ \begin{array}{l} \boxed{S(f,P) = \sum_{i=1}^{n} M(f)\, \Delta x_i} \\ \boxed{s(f,P) = \sum_{i=1}^{n} m(f)\, \Delta x_i} \end{array} \right.$ y $s(f,P) \leq \underset{\substack{\downarrow \\ f \text{ continua}}}{\int} f \leq S(f,P)$

Entonces: $S(f,P) = \sum_{i=1}^{n} f(x_{i-1})\, \Delta x_i = \sum_{i=1}^{2} e^{-\left(\frac{i-1}{2}\right)^2} \frac{1}{2} = e^{-\left(\frac{\cancel{1-1}\,0}{2}\right)^2} \frac{1}{2} + e^{-\left(\frac{2-1}{2}\right)^2} \frac{1}{2} = \frac{1}{2} + \frac{1}{2} e^{-\frac{1}{4}}$

$= \frac{1}{2}\left(1 + e^{-\frac{1}{4}}\right)$

$s(f,P) = \sum_{i=1}^{n} f(x_i)\, \Delta x_i = \sum_{i=1}^{2} e^{-\left(\frac{i}{n}\right)^2} \frac{1}{2} = e^{-\left(\frac{1}{2}\right)^2} \frac{1}{2} + e^{-\left(\frac{2}{2}\right)^2} \frac{1}{2} = \frac{1}{2}\left(e^{-\frac{1}{4}} + e^{-1}\right)$

Como $\quad r(f,P) \leq \int_0^1 e^{-x^2}dx \leq S(f,P)$

$$\Leftrightarrow \frac{1}{2}\left(e^{-\frac{1}{4}} + e^{-1}\right) \leq \int_0^1 e^{-x^2}dx \leq \frac{1}{2}\left(1+e^{-\frac{1}{4}}\right)$$

$$\Leftrightarrow \frac{1}{2}\left(\frac{1}{e^{\frac{1}{4}}} + \frac{1}{e}\right) \leq \int_0^1 e^{-x^2}dx \leq \frac{1}{2}\left(1+\frac{1}{e^{\frac{1}{4}}}\right)$$

$$\frac{1}{2}\left(\frac{1}{e^{\frac{1}{4}}} + \frac{1}{e}\right) \leq \int_0^1 e^{-x^2}\,dx \leq \frac{1}{2}\left(1+\frac{1}{e^{\frac{1}{4}}}\right)$$

Calcular integrales es mucho aprendizaje propio... ejerciten!

Quedo atenta a dudas :)

Éxito en su estudio

-Bianca.

En el curso Electromagnetismo verán que aparecen integrales como las que vimos en (P1) f) g) h)...
Así que son muy útiles!!