# DESARROLLO AUX 8

## SUMATORIAS

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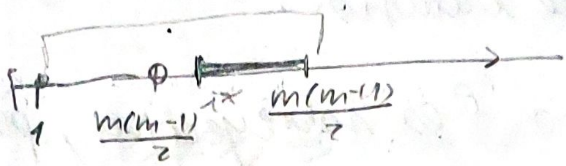
# Auxiliar 8: Sumatorias

**P1**

a) $$\sum_{i=\frac{m(m-1)}{2}+1}^{\frac{m(m+1)}{2}} (2i-1)$$ si $m=0 \Rightarrow i=1$

lo menor posible

$\frac{m(m-1)}{2}+1 \leq i^*$




La idea es reescribir el dominio de la suma:

$$\left[\frac{m(m-1)}{2}+1, \frac{m(m+1)}{2}\right] = \left[1, \frac{m(m+1)}{2}\right] \setminus \left[1, \frac{m(m-1)}{2}\right]$$

$\frac{m(m+1)}{2} =: \alpha$ $\quad$ $\frac{m(m-1)}{2} =: \beta$

$$= \sum_{i=1}^{\alpha} (2i-1) - \sum_{i=1}^{\beta} (2i-1)$$

$$= 2\sum_{i=1}^{\alpha} i - \boxed{\sum_{i=1}^{\alpha} 1} - \left(2\boxed{\sum_{i=1}^{\beta} i} - \sum_{i=1}^{\beta} 1\right)$$

Suma de Gauss $\quad$ Suma de 1

$$= 2\,\frac{\alpha(\alpha+1)}{2} - \boxed{(\alpha-1+1)} - 2\,\boxed{\frac{\beta(\beta+1)}{2}} + (\beta-1+1)$$

$$= \alpha(\alpha+1) - \alpha - \beta(\beta+1) + \beta$$

$$= \alpha^2 \cancel{+\alpha} \cancel{-\alpha} - \beta^2 \cancel{-\beta} \cancel{+\beta} = \alpha^2 - \beta^2$$

$$= \left(\frac{m(m+1)}{2}\right)^2 - \left(\frac{m(m-1)}{2}\right)^2 \qquad m^2\left((m+1)^2 + (m-1)^2\right) = m^2(m^2+2)$$

$$= \frac{m^2(m+1)^2}{4} - \frac{m^2(m-1)^2}{4} = \frac{m^2\left((m+1)^2 - (m-1)^2\right)}{4}$$

$$= \frac{m^2(4m)}{4} = m^3. \blacksquare$$

P1 b) $$\text{☺} := \sum_{k=1}^{n} \frac{1}{\sqrt{k(k+1)}\left(\sqrt{k+1}+\sqrt{k}\right)}$$ → Racionalizar!

suena a telescópica

$$\underbrace{\frac{1}{\sqrt{k(k+1)}\left(\sqrt{k+1}+\sqrt{k}\right)}}_{=\,(\star)} = \frac{1}{\sqrt{k(k+1)}}\cdot\frac{1}{\sqrt{k+1}+\sqrt{k}}$$

$$= \frac{1}{\sqrt{k+1}+\sqrt{k}}\cdot\frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k+1}-\sqrt{k}}$$

$$= \frac{\sqrt{k+1}-\sqrt{k}}{\cancel{k}+1\cancel{-k}} = \sqrt{k+1}-\sqrt{k}$$

$$\Rightarrow (\star) = \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k(k+1)}} = \frac{\cancel{\sqrt{k+1}}}{\sqrt{k}\,\cancel{\sqrt{k+1}}} - \frac{\cancel{\sqrt{k}}}{\cancel{\sqrt{k}}\,\sqrt{k+1}} = \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}$$

Separan fracciones

telescópica!

Luego, la suma es:

$$\text{☺} = \sum_{k=1}^{n}\left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}\right) = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{n+1}} \quad \blacksquare$$

P1) c)

$$\sum_{k=0}^{n} k\,7^{k}\binom{n}{k}$$ → screams binomio de Newton

En efecto,

término 0-ésimo es cero

$$\sum_{k=0}^{n} k\,7^{k}\binom{n}{k} = \sum_{k=1}^{n} k\,7^{k}\frac{n!}{(n-k)!\,k!} = \sum_{k=1}^{n} \cancel{k}\,7^{k}\frac{n!}{(n-k)!\,\cancel{k}\,(k-1)\cdots 1}$$

$$= \sum_{k=1}^{n} 7^{k}\frac{n!}{(n-k)!\,(k-1)!} = \sum_{k=1}^{n} 7^{k}\frac{n\,(n-1)!}{(n-k)!\,(k-1)!} \qquad \frac{(n-1)!}{(n-k)!\,(k-1)!} = \binom{n-1}{k-1}$$

$$= n\sum_{k=1}^{n} 7^{k}\binom{n-1}{k-1} = n\sum_{k=0}^{n-1} 7^{k+1}\binom{n-1}{k}\cdot \underbrace{1^{n-1-k}}_{=1}$$

cambio de índice

$$= n7\sum_{k=0}^{n-1} 7^{k}\binom{n-1}{k}1^{n-1-k} = 7n(7+1)^{n-1} = 7n8^{n-1} \qquad \blacksquare$$

$\sum_{i=1}^{n} i2^i$ (Pi) d)

telescópica

Notar que $\sum_{i=1}^{n}\left[(i+1)2^{i+1} - i2^i\right] \overset{!}{=} (n+1)2^{n+1} - 2$

$\iff \sum_{i=1}^{n}\left[i2^{i+1} + 2^{i+1} - i2^i\right] = \underbrace{(n+1)2^{n+1} - 2}_{=:\mu}$

$= i(2^{i+1} - 2^i)$

$= i2^i(2-1)$

$= i2^i \cdot 1$

$= \boxed{i2^i}$

$\iff \sum_{i=1}^{n} \boxed{i2^i} + 2^{i+1} = \mu$

geométrica: $2\sum_{i=1}^{n} 2^i = 2\cdot 2^1 \frac{2^{n-1+1} - 1}{2-1}$ "Uno"

$= 4(2^n - 1)$

$= 2^2\cdot 2^n - 2^2$

$= 2^{n+2} - 4$

$\iff \sum_{i=1}^{n} i2^i = \mu - \sum_{i=1}^{n} 2^{i+1}$

$= (n+1)2^{n+1} - 2 - \left[2^{n+2} - 4\right]$

$= 2^{n+1}\left((n+1) - 2\right) - 2 + 4$

$\iff \sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2$ ∎

$n \geq 1$. $r \neq 1$.

(P2) $S_n := \sum_{k=1}^{n} k r^k$. P.D.Q. $S_n = r(S_n - nr^n) + \sum_{k=0}^{n-1} r^{k+1}$

En efecto,

$$S_n = \sum_{k=1}^{n} k r^k = \sum_{k=0}^{n-1} (k+1) r^{k+1} = \sum_{k=0}^{n-1} \left(k r^{k+1} + r^{k+1}\right)$$

cambio índice $k \to 0$ (fuera)

$$= \sum_{k=0}^{n-1} k r^{k+1} + \sum_{k=0}^{n-1} r^{k+1}$$

cero conveniente

$$= \sum_{k=0}^{n-1} k r^{k+1} + nr^{n+1} - nr^{n+1} + \sum_{k=0}^{n-1} r^{k+1}$$

$$= \sum_{k=0}^{n} k r^{k+1} - nr^{n+1} + \sum_{k=0}^{n-1} r^{k+1}$$

$$= r \sum_{k=0}^{n} k r^{k} - nr^{n+1} + \sum_{k=0}^{n-1} r^{k+1}$$

$$= r S_n - nr^{n+1} + \sum_{k=0}^{n-1} r^{k+1}$$

$$= r(S_n - nr^n) + \sum_{k=0}^{n-1} r^{k+1}. \blacksquare$$

(P2) ii)

P.D.Q. $S_n = \dfrac{r-(n+1)r^{n+1}+nr^{n+2}}{(1-r)^2}$

De la parte anterior, se tiene que:

$$S_n = r(S_n - nr^n) + \sum_{k=0}^{n-1} r^{k+1}$$

$$= r(S_n - nr^n) + r\sum_{k=0}^{n-1} r^k \quad \Big\} \; r \neq 1$$

$$= r(S_n - nr^n) + r\left(r^0 \frac{r^{(n-1)+1-0}-1}{r-1}\right)$$

$$= r(S_n - nr^n) + r\left(\frac{r^n - 1}{r-1}\right)$$

$$= r(S_n - nr^n) + \left(\frac{r^{n+1} - r}{r-1}\right)$$

$$\Leftrightarrow S_n(1-r) = -nr^{n+1} + \frac{r^{n+1}-r}{r-1}$$

$$= \frac{(r-1)(-nr^{n+1}) + r^{n+1} - r}{r-1}$$

$$= \frac{-nr^{n+2} + \underbrace{nr^{n+1} + r^{n+1}}_{=(n+1)r^{n+1}} - r}{r-1}$$

$$\Leftrightarrow S_n = \frac{1}{(1-r)} \cdot \frac{-nr^{n+2} + nr^{n+1} + r^{n+1} - r}{r-1}$$

$$\Leftrightarrow S_n = \frac{r-(n+1)r^{n+1}+nr^{n+2}}{(1-r)^2}$$

$\boxtimes$

(12) iii)
Calcular $S_n\big|_{r=1}$ con $S_n = \sum_{k=1}^{n} k r^k$

La fórmula recursiva de recién no sirve pues es para $r \neq 1$.

Si $S_n = \sum_{k=1}^{n} k r^k \underset{r=1}{\Rightarrow} \sum_{k=1}^{n} k\, 1^k \underset{1^k = 1\ (\forall k \in \mathbb{N})}{=} \sum_{k=1}^{n} k \underset{\text{Gauss.}}{=} \frac{n(n+1)}{2}$ ☑

(P3) i)

P.D.Q. $(\forall n,k\in\mathbb{N}), k<n, : \binom{n}{k-1} = \frac{k}{n-k+1}\binom{n}{k}$

Sean $n,k\in\mathbb{N}$ t.q. $k<n$.

En efecto,

$$\binom{n}{k-1} = \frac{n!}{[n-(k-1)]!\,(k-1)!}$$

$$= \frac{n!}{(n-k+1)!\,(k-1)!}$$

$$= \frac{n!}{\boxed{(n-k+1)}(n-k)(n-k-1)\cdots 1\,(k-1)!}$$

$$= \frac{n!}{\boxed{(n-k+1)}\cdot(n-k)!\,\underbrace{(k-1)!\cdot k}_{k!}}\cdot k$$

uno conveniente

$$= \frac{k}{n-k+1}\,\frac{n!}{(n-k)!\,k!}$$

$$= \frac{k}{n-k+1}\binom{n}{k} \quad \blacksquare$$

(P3) ii) Calcular $\sum_{j=1}^{n} \frac{n-j+1}{j}\binom{n}{j-1}$

Por la identidad ya vista:

$$\sum_{j=1}^{n} \frac{n-j+1}{j}\binom{n}{j-1} = \sum_{j=1}^{n} \frac{\cancel{n-j+1}}{\cancel{j}} \cdot \frac{\cancel{j}}{\cancel{n-j+1}}\binom{n}{j}$$

$$= \sum_{j=1}^{n}\binom{n}{j} = \sum_{j=1}^{n}\binom{n}{j} + \underbrace{\binom{n}{0} - \binom{n}{0}}_{\text{cero conveniente}}$$

$$= \left[\sum_{j=0}^{n}\binom{n}{j}\right] - 1 = \left[\sum_{j=0}^{n} 1^{n} 1^{n-k}\binom{n}{g}\right] - 1 = (1+1)^{n} - 1$$

$$= 2^{n} - 1$$

(P4) a) P.D.Q. $\sum_{i=1}^{2^k} \frac{1}{2^k+i} \leq 1 \quad (\forall k \in \mathbb{N})$

En efecto,

$$\sum_{i=1}^{2^k} \frac{1}{2^k+i} = \frac{1}{2^k+1} + \frac{1}{2^k+2} + \frac{1}{2^k+3} + \dots + \frac{1}{2^k+2^k}$$

$$\leq \underbrace{\frac{1}{2^k} + \frac{1}{2^k} + \frac{1}{2^k} + \dots + \frac{1}{2^k}}_{2^k \text{ veces}} = 2^k \frac{1}{2^k} = 1$$

pues $\alpha \geq \beta \Rightarrow \underbrace{\frac{1}{\alpha}}_{2^k+i=\alpha} \leq \underbrace{\frac{1}{\beta}}_{\beta=2^k}$

$$\Rightarrow \sum_{i=1}^{2^k} \frac{1}{2^k+i} \leq 1. \quad \blacksquare$$

P4 b): P.D.Q. $\sum_{j=0}^{n} \binom{j+3}{3} = \binom{n+4}{4}$ .

Identidad de Pascal $\equiv \binom{a}{3} = \binom{a+1}{4} - \binom{a}{4}$ , $a \geq 4$

En efecto,

$$\sum_{j=0}^{n} \binom{j+3}{3} = \sum_{j=3}^{n+3} \binom{j}{3} = \overbrace{\binom{3}{3}}^{\text{4-ésimo término}} + \sum_{j=4}^{n+3} \binom{j}{3}$$

$$\underset{\text{Pascal}}{=} 1 + \sum_{j=4}^{n+3} \left[ \binom{j+1}{4} - \binom{j}{4} \right] \text{ telescópica}$$

$$= 1 + \binom{(n+3)+1}{4} - \cancel{\binom{4}{4}}^{1}$$

$$= 1 + \binom{n+4}{4} - 1 = \binom{n+4}{4} . \ \blacksquare$$

(P4) c) P.D.Q. $\sum_{k=0}^{n}(1-x)^k = \sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}x^k$ $(\forall n \in \mathbb{N})$

Supongamos que $1-x \neq 1$

$$\Rightarrow \sum_{k=0}^{n}(1-x)^k = (1-x)^0 \frac{(1-x)^{n-0+1}-1}{(1-x)-1}$$

$$= \frac{(1-x)^n - 1}{-x} = \frac{1-(1-x)^n}{x}$$ ........ Newton! (Binomio)

$$= \frac{1-\sum_{k=0}^{n+1} 1^{(n+1)-k}(-x)^k\binom{n+1}{k}}{x}$$

0-ésimo término

$$= \frac{1-\left(\left[\sum_{k=1}^{n+1} \cancel{1^{(n+1)-k}}^{\,1}(-x)^k\binom{n+1}{k}\right]+\cancel{1}\right)}{x}$$

$$= \frac{-1}{-x}\sum_{k=1}^{n+1}(-x)^k\binom{n+1}{k} = -\sum_{k=1}^{n+1}(-x)^{k+1}\binom{n+1}{k}$$

$$= \sum_{k=0}^{n}(-x)^k\binom{n+1}{k+1} = \sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}x^k. \blacksquare$$

P4 d) $p, q \in \mathbb{R}$ tq. $p+q=1$.

q $\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} k^2$

Notar que $k^2 = k^2 - k + k$
$= k(k-1) + k$

$$\Rightarrow \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} k^2 = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \left[k(k-1) + k\right]$$

$$= \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} k(k-1) + \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} k$$

excluimos términos nulos

$$= \sum_{k=2}^{n} \binom{n}{k} p^k q^{n-k} k(k-1) + \sum_{k=1}^{n} \binom{n}{k} p^k q^{n-k} k$$

Ojo: $* \binom{n}{k} k(k-1) = \frac{n!}{(n-k)!\, k!} k(k-1) = \frac{n(n-1)(n-2)!}{(n-k)!\,(k-2)!} = n(n-1)\binom{n-2}{k-2}$

$* \binom{n}{k} k = \frac{n!}{(n-k)! \cdot k!} k = \frac{n(n-1)!}{(n-k)! \cdot (k-1)!} = n\binom{n-1}{k-1}$

$$= n(n-1) \sum_{k=2}^{n} \binom{n-2}{k-2} p^k q^{n-k} + n \sum_{k=1}^{n} \binom{n-1}{k-1} p^k q^{n-k}$$

índices

$$= n(n-1) \sum_{k=0}^{n-2} \binom{n-2}{k} p^{k+2} q^{n-(k+2)} + n \sum_{k=0}^{n-1} \binom{n-1}{k} p^{(k+1)} q^{n-(k+1)}$$

reordenar

$$= n(n-1) p^2 \sum_{k=0}^{n-2} \binom{n-2}{k} p^k q^{(n-2)-k} + np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k q^{(n-1)-k}$$

$$= n(n-1)p^2 \underbrace{(p+q)}_{=1}{}^{n-2} + np\cancel{\underbrace{(p+q)}_{=1}{}^{n-1}}$$ ← Binomio de Newton

$$= n(n-1)p^2 + np$$

(P4) e) P.D.Q. $\displaystyle\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k+1}=\frac{1}{n+1}$

En efecto,

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k+1}=\sum_{k=0}^{n}\frac{n!\,(n+1)}{(n-k)!\cdot k!\,(k+1)}(-1)^k\frac{1}{(n+1)}$$

$$=\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}(-1)^k(-1)\cdot\frac{1}{(-1)}$$

$$=-\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}(-1)^{k+1}$$

$$=-\frac{1}{n+1}\left(\sum_{k=1}^{n+1}\binom{n+1}{k}(-1)^k\right)$$

$$=-\frac{1}{n+1}\left(\left[\sum_{k=1}^{n+1}\binom{n+1}{k}(-1)^k\right]+\binom{n+1}{0}(-1)^0-\overbrace{\cancel{\binom{n+1}{0}(-1)^0}}^{1}\right)$$

$$=-\frac{1}{n+1}\left(\sum_{k=0}^{n+1}\binom{n+1}{k}1^{(n+1)-k=1}(-1)^k-1\right)$$

$$=-\frac{1}{n+1}\left(\overbrace{\cancel{(1-1)}}^{0}{}^{n+1}-1\right)=\frac{1}{n+1}\ \blacksquare$$

P4) f) $\displaystyle\sum_{k=1}^{n} \frac{1}{k}\binom{n}{k-1}5^{k}$

$$\frac{1}{k}\binom{m-1}{j-1} = \frac{1}{m}\binom{m}{j} \quad \forall m\in\mathbb{N},\ m\geq 1,\ j\in[1..m]$$

En efecto, aplicando la indicación:

$$\sum_{k=1}^{n}\frac{1}{k}\binom{n}{k-1}5^{k} = \sum_{k=1}^{n}\frac{1}{n+1}\binom{n+1}{k}5^{k}$$

$$= \frac{1}{n+1}\left(\sum_{k=1}^{n}\binom{n+1}{k}5^{k} + \underbrace{\binom{n+1}{0}5^{0}}_{0\text{-ésimo término}} + \underbrace{\binom{n+1}{n+1}5^{n+1}}_{(n+1)\text{-ésimo término}} - \overset{1}{\cancel{\binom{n+1}{0}}}5^{0} - \overbrace{\binom{n+1}{n+1}5^{n+1}}^{5^{n-1}}\right)$$

$$= \frac{1}{n+1}\left(\left[\sum_{k=0}^{n+1}\binom{n+1}{k}5^{k}\right] - 1 - 5^{n+1}\right)$$

$$= \frac{1}{n+1}\left((1+5)^{n+1} - 1 - 5^{n+1}\right)$$

$$= \frac{1}{n+1}\left(6^{n+1} - 5^{n+1} - 1\right). \quad \boxtimes$$

P5 a) P.D.Q. $(\forall n,i,k\in\mathbb{N}),\ k\leq i\leq n:\ \binom{n}{i}\binom{i}{k}=\binom{n}{k}\binom{n-k}{i-k}$

En efecto,

$$\binom{n}{i}\binom{i}{k}=\frac{n!}{(n-i)!\,\cancel{i!}}\cdot\frac{\cancel{i!}}{(i-k)!\,k!}\ \ldots\ \text{def}\binom{\cdot}{\cdot}.$$

uno conveniente

$$=\frac{n!}{(n-k)!\,k!}\cdot\frac{(n-k)!}{(n-i)!\,(i-k)!}$$

$$=\binom{n}{k}\binom{n-k}{i-k}\ \ldots\ \text{def}\binom{\cdot}{\cdot}.\ \blacksquare$$

P5 b) P.D.Q. $\sum_{i=k}^{n}\binom{n}{i}\binom{i}{k}=\binom{n}{k}2^{n-k}$ $\quad\binom{n}{k}\neq\binom{n}{k}(i)$

En efecto, $\sum_{i=k}^{n}\binom{n}{i}\binom{i}{k}\overset{a)}{=}\sum_{i=k}^{n}\binom{n}{k}\binom{n-k}{i-k}\overset{!}{=}\binom{n}{k}\sum_{i=k}^{n}\binom{n-k}{i-k}$

$$=\binom{n}{k}\sum_{i=0}^{n-k}\binom{n-k}{i}=\binom{n}{k}\sum_{i=0}^{n-k}\binom{n-k}{i}1^{(n-k)-i}\cdot 1^{i}$$

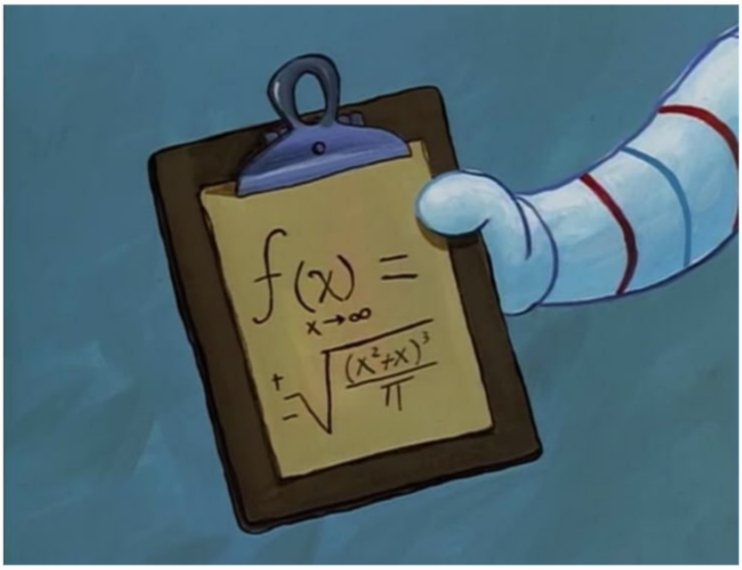
cambio de índices

$$=\binom{n}{k}(1+1)^{n-k}=\binom{n}{k}2^{n-k}.\ \blacksquare$$

P5 c) Calcular $\sum_{k=0}^{n}\left[\sum_{i=k}^{n}\binom{n}{i}\binom{i}{k}\right]=\sum_{k=0}^{n}\binom{n}{k}2^{n-k}\cdot 1^{k}=(2+1)^{n}=3^{n}.\ \blacksquare$

uno conveniente

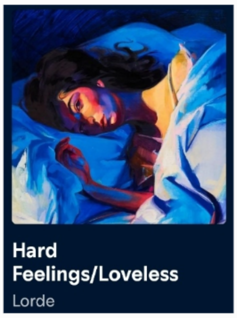
Las sumatorias son una herramienta fundamental detrás de diversos modelamientos.
Es muy valioso y útil manejarse con ellas!!




Espero que les haya gustado el auxiliar, donde vimos los distintos "trukitos" que son buenos de conocer.

Siempre atenta a sus dudas :)

Éxito en su estudio!!!

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