3. Sea una curva parametrizada en **coordenadas polares** por:

$$\vec{r} = r(\theta)\hat{r}$$

La curvatura κ de la curva estará dada por la siguiente expresión:

$$\kappa = \frac{1}{\rho_c} = \frac{||\dot{\vec{r}}(\theta) \times \ddot{\vec{r}}(\theta)||}{||\dot{\vec{r}}(\theta)||^3}$$

a) Calcule la curvatura κ de la curva utilizando esta parametrización en función de $\frac{dr}{d\theta}$ y $\frac{d^2 r}{d\theta^2}$

b) Calcule la curvatura del "caracol de pascal", dado por:

$$r(\theta) = a + b\cos(\theta)$$

a) En coordenadas **polares**, cualquier posición puede ser descrita como:

$$\vec{r} = r\hat{r}$$

Para calcular la curvatura, se calcula la velocidad y aceleración teniendo en cuenta lo siguiente:

$$\hat{r} = \cos(\theta)\hat{x} + \sin(\theta)\hat{y} \qquad \Rightarrow \qquad \dot{\hat{r}} = -\sin(\theta)\dot{\theta}\hat{x} + \cos(\theta)\dot{\theta}\hat{y} = \dot{\theta}\hat{\theta} \quad +0,3$$

$$\hat{\theta} = -\sin(\theta)\hat{x} + \cos(\theta)\hat{y} \qquad \qquad \dot{\hat{\theta}} = -\cos(\theta)\dot{\theta}\hat{x} - \sin(\theta)\dot{\theta}\hat{y} = -\dot{\theta}\hat{r} \quad +0,3$$

$$\dot{\vec{r}} = \frac{d}{dt}(r\hat{r}) = \dot{r}\hat{r} + r\dot{\hat{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \quad +0,4 \text{ con/sin reemplazar } \dot{\hat{r}}$$

$$\ddot{\vec{r}} = \frac{d}{dt}(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) = \ddot{r}\hat{r} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\theta}\hat{\theta} + r(\ddot{\theta}\hat{\theta} + \dot{\theta}\dot{\hat{\theta}})$$

$$= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r}$$

$$= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} \quad +0,4$$

con/sin reemplazar $\dot{r}$ y $\ddot{r}$

Se calculan numerador y denominador de la expresión de la curvatura: 

$$|\dot{\vec{r}} \times \ddot{\vec{r}}| = |(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) \times ((\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta})|$$

$$= |\dot{r}(r\ddot{\theta} + 2\dot{r}\dot{\theta})(\hat{r}\times\hat{\theta}) + r\dot{\theta}(\ddot{r} - r\dot{\theta}^2)(\hat{\theta}\times\hat{r})|$$

$$= |(r\dot{r}\ddot{\theta} + 2\dot{r}^2\dot{\theta} - r\ddot{r}\dot{\theta} + r^2\dot{\theta}^3)\hat{z}|$$

$$= r\dot{r}\ddot{\theta} + 2\dot{r}^2\dot{\theta} - r\ddot{r}\dot{\theta} + r^2\dot{\theta}^3 \quad +0,3$$

con/sin reemplazar $\dot{r}$ y $\ddot{r}$

$$|\dot{\vec{r}}|^3 = |\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}| = (\dot{r}^2 + r^2\dot{\theta}^2)^{3/2} \quad +0,3 \text{ con/sin reemplazar } \dot{\hat{r}}$$

Con esto, la curvatura será:

$$K = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3} = \frac{r\dot{r}\ddot{\theta} + 2\dot{r}^2\dot{\theta} - r\ddot{r}\dot{\theta} + r^2\dot{\theta}^3}{(\dot{r}^2 + r^2\dot{\theta}^2)^{3/2}} \quad +0{,}5$$

con/sin reemplazar $\dot{r}$ y $\ddot{r}$

La curva trabajada está parametrizada por $\vec{r} = r(\theta)\hat{r}$. Con esto:

$$\dot{r}(\theta) = \frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt} = \frac{dr}{d\theta}\dot{\theta} = r'\dot{\theta} \qquad +0{,}5$$

$$\ddot{r}(\theta) = \frac{d^2r}{dt^2} = \frac{d}{dt}(r'\dot{\theta}) = \left(\frac{dr'}{d\theta}\cdot\frac{d\theta}{dt}\right)\dot{\theta} + r'\ddot{\theta} = \frac{d^2r}{d\theta^2}\dot{\theta}^2 + \frac{dr}{d\theta}\ddot{\theta} = r''\dot{\theta}^2 + r'\ddot{\theta} \qquad +0{,}5$$

Reemplazando:

$$K = \frac{r\cdot r'\dot{\theta}\cdot\ddot{\theta} + 2\cdot r'^2\dot{\theta}^2\cdot\dot{\theta} - r(r''\dot{\theta}^2 + r'\ddot{\theta})\dot{\theta} + r^2\dot{\theta}^3}{(r'^2\dot{\theta}^2 + r^2\dot{\theta}^2)^{3/2}}$$

$$= \frac{\cancel{rr'\dot{\theta}\ddot{\theta}} + 2r'^2\dot{\theta}^3 - rr''\dot{\theta}^3 - \cancel{rr'\dot{\theta}\ddot{\theta}} + r^2\dot{\theta}^3}{(r'^2\dot{\theta}^2 + r^2\dot{\theta}^2)^{3/2}}$$

$$= \frac{\cancel{\dot{\theta}^3}(2r'^2 - rr'' + r^2)}{\cancel{\dot{\theta}^3}(r'^2 + r^2)^{3/2}}$$

$$= \frac{2r'^2 - rr'' + r^2}{(r'^2 + r^2)^{3/2}} \qquad +0{,}5 \text{ curvatura}$$

b) Para calcular la curvatura de la curva dada, se calculan las derivadas necesarias para evaluar la expresión:

$$r = a + b\cos(\theta)$$

$$r' = \frac{dr}{d\theta} = -b\sin(\theta) \qquad +0{,}5$$

$$r'' = \frac{d^2r}{d\theta^2} = -b\cos(\theta) \qquad +0{,}5$$

Con esto, la curvtura será:

$$K = \frac{2b^2\sin^2(\theta) - (a + b\cos(\theta))(-b\cos(\theta)) + (a + b\cos(\theta))^2}{(b^2\sin^2(\theta) + (a + b\cos(\theta))^2)^{3/2}} \qquad +0{,}5 \text{ reemplazar}$$

$$= \frac{2b^2\sin^2(\theta) + ab\cos(\theta) + b^2\cos^2(\theta) + a^2 + 2ab\cos(\theta) + b^2\cos^2(\theta)}{(b^2\sin^2(\theta) + a^2 + 2ab\cos(\theta) + b^2\cos^2(\theta))^{3/2}}$$

$$= \frac{2b^2(\sin^2(\theta) + \cos^2(\theta)) + 3ab\cos(\theta) + a^2}{(b^2(\sin^2(\theta) + \cos^2(\theta)) + 2ab\cos(\theta) + a^2)^{3/2}}$$

$$= \frac{2b^2 + 3ab\cos(\theta) + a^2}{(b^2 + 2ab\cos(\theta) + a^2)^{3/2}}$$ +0,5