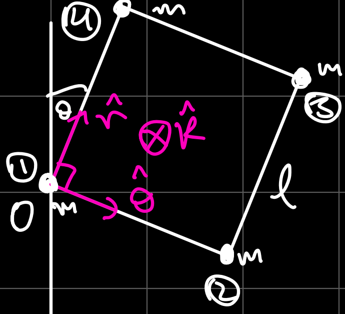
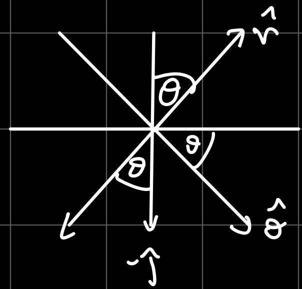
a)





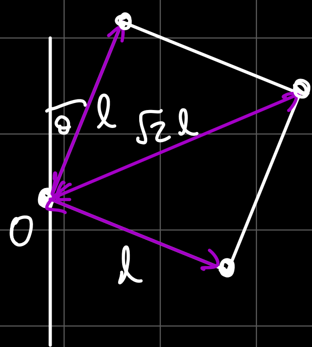
Se pide:

$$\vec{L}_0 = I_0 \vec{\Omega} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot\theta \end{bmatrix} = \overbrace{\begin{pmatrix} I_{xz}\dot\theta \\ I_{yz}\dot\theta \\ I_{zz}\dot\theta \end{pmatrix}}^{I_{yz}=I_{xz}=0 \text{ pues } z=0} = I_{zz}\dot\theta \hat{k}$$

Calculemos I_{zz}:

2 formas:

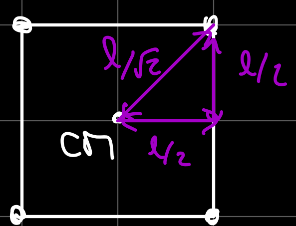
- Directo:



$$I_{zz} = \sum_i m\overbrace{(x_i^2+y_i^2)}^{r^2}$$

$$= ml^2 + m\cdot 2l^2 + ml^2 = 4ml^2$$

- En el CM y usamos Steiner:



Usando simetría:

$$I_{zz,cm} = \sum_i m(x_i^2+y_i^2) = 4\cdot m\cdot \frac{l^2}{2} = 2ml^2$$

Con steiner lo sacamos en una esquina:

$$I_{zz} = I_{zz,cm} + 4m(x_{cm}^2+y_{cm}^2)$$

$$= 2ml^2 + 4m\frac{l^2}{2} = 4ml^2$$

Luego: $\boxed{\vec{L}_0 = 4ml^2\dot\theta\hat{k}}$

b) Forma 1:

$$\vec{\tau}_0 = \vec{\tau}_{mg} + \vec{\tau}_{F_{eje}}$$

$$= \vec{\tau}_{mg,1} + \vec{\tau}_{mg,2} + \vec{\tau}_{mg,3} + \tau_{mg,4} + \vec{\tau}_{F_{eje}}$$

$$= \vec{r}_1 \times m\vec{g} + \vec{r}_2 \times m\vec{g} + \vec{r}_3 \times m\vec{g} + r_4 \times m\vec{g} + \vec{r}_1 \times \vec{F}_{eje}$$

donde: $\vec{r}_1 = 0 \quad \vec{r}_2 = l\hat{\theta} \quad \vec{r}_3 = l\hat{\theta} + l\hat{r} \quad \vec{r}_4 = l\hat{r}$

$$m\vec{g} = mg(-\hat{j}) = mg(-\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$$

Luego: $\vec{r}_1 \times m\vec{g} = 0$

$$\vec{r}_2 \times m\vec{g} = l\hat{\theta} \times mg(-\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) = mgl\cos\theta\,\hat{k}$$

$$\vec{r}_3 \times m\vec{g} = l(\hat{\theta} + \hat{r}) \times mg(-\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) = +mgl\cos\theta\,\hat{k} + mgl\sin\theta\,\hat{k}$$

$$\vec{r}_4 \times m\vec{g} = l\hat{r} \times mg(-\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) = mgl\sin\theta\,\hat{k}$$

$$\vec{r}_1 \times \vec{F}_{eje} = 0$$

$$\therefore \boxed{\vec{\tau}_0 = 2mgl(\sin\theta + \cos\theta)\,\hat{k}}$$

Forma 2: El peso actua sobre el CM

$$\Rightarrow \vec{\tau}_0 = \frac{l}{2}(\hat{r} + \hat{\theta}) \times 4mg(-\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) = 2mgl(\sin\theta + \cos\theta)\,\hat{k}$$

Ec. de momentum angular y torque:

$$\dot{L}_0 = \vec{\tau}_0$$

$$\Rightarrow 4ml^2\ddot{\theta}\,\hat{k} = 2mgl(\cos\theta + \sin\theta)\,\hat{k}$$

$$\Rightarrow \boxed{\ddot{\theta} = \frac{g}{2l}(\cos\theta + \sin\theta)} \quad (*)$$

c) $M\vec{A}_{cm} = \sum \vec{F}_{ext}$

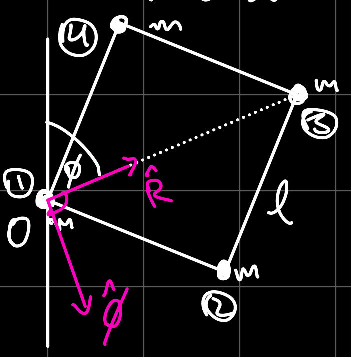
$$\vec{R}_{cm} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \frac{m\cdot 0 + m l\hat{\theta} + m(l\hat{\theta} + l\hat{r}) + m l\hat{r}}{4m} = \frac{l}{2}(\hat{r} + \hat{\theta})$$

$$\vec{V}_{cm} = \frac{l}{2}(\dot{\theta}\hat{\theta} - \dot{\theta}\hat{r})$$

$$\vec{A}_{cm} = \frac{l}{2}(\ddot{\theta}\hat{\theta} - \dot{\theta}^2\hat{r} - \ddot{\theta}\hat{r} - \dot{\theta}^2\hat{\theta}) = \frac{l}{2}\left[(\ddot{\theta} - \dot{\theta}^2)\hat{\theta} - (\ddot{\theta} + \dot{\theta}^2)\hat{r}\right]$$

$$4m\vec{A}_{cm} = \vec{F}_{eje} + 4m\vec{g} \quad \Rightarrow \vec{F}_{eje} = 4m\left[\vec{A}_{cm} - \vec{g}\right]$$

Más fácil:



$\phi = \theta + 45° \Rightarrow \dot{\phi} = \dot{\theta}\,,\ \ddot{\phi} = \ddot{\theta}$

$\Rightarrow \vec{R}_{cm} = \frac{l}{\sqrt{2}}\hat{R} \qquad \vec{V}_{cm} = \frac{l}{\sqrt{2}}\dot{\phi}\hat{\phi}$

$\vec{A}_{cm} = \frac{l}{\sqrt{2}}(\ddot{\phi}\hat{\phi} - \dot{\phi}^2\hat{R})$

$\Rightarrow \vec{F}_{eje} = 4m\vec{A}_{cm} - 4m\vec{g}\,,\quad \vec{g} = g(-\cos\phi\,\hat{R} + \sin\phi\,\hat{\phi})$

De $(*)$: $\dot{\theta}\,d\dot{\theta} = \frac{g}{2l}(\cos\theta + \sin\theta)\,d\theta \Big/ \int_0^{\dot{\theta}}, \int_0^{\theta}$

$$\Rightarrow \frac{\dot{\theta}^2}{2} = \frac{g}{2l}(\sin\theta - \cos\theta)\Big|_0^{\theta} = \frac{g}{2l}(\sin\theta - \cos\theta + 1)$$