Sistemas de Coordenadas

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1. Coordenadas Cartesianas

1.1. Vectores base $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{k}}\}$ y sus derivadas temporales:

$$\hat{\boldsymbol{x}} = \hat{\boldsymbol{\imath}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \hat{\boldsymbol{y}} = \hat{\boldsymbol{\jmath}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \hat{\boldsymbol{k}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\dot{\hat{\boldsymbol{x}}} = \dot{\hat{\boldsymbol{\imath}}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \dot{\hat{\boldsymbol{y}}} = \dot{\hat{\boldsymbol{\jmath}}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \dot{\hat{\boldsymbol{k}}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

1.2. Coordenadas:

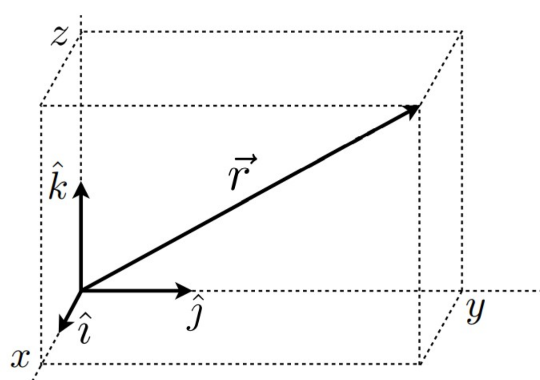
$$x \in (-\infty, \infty), \qquad y \in (-\infty, \infty), \qquad z \in (-\infty, \infty)$$

1.3. Vectores Cinemáticos:

Posición: $\vec{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{k}}$

Velocidad: $\vec{v} = \dot{x}\hat{\boldsymbol{x}} + \dot{y}\hat{\boldsymbol{y}} + \dot{z}\hat{\boldsymbol{k}}$

Aceleración: $\vec{a} = \ddot{x}\hat{\boldsymbol{x}} + \ddot{y}\hat{\boldsymbol{y}} + \ddot{z}\hat{\boldsymbol{k}}$



2. Coordenadas Cilíndricas

2.1. Vectores base $\{\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{k}}\}$ y sus derivadas temporales:

$$\hat{\boldsymbol{\rho}} = \cos\phi\hat{\boldsymbol{x}} + \sin\phi\hat{\boldsymbol{y}}, \qquad \hat{\boldsymbol{\phi}} = -\sin\phi\hat{\boldsymbol{x}} + \cos\phi\hat{\boldsymbol{y}}, \qquad \hat{\boldsymbol{k}} = \hat{\boldsymbol{k}}$$

$$\dot{\hat{\boldsymbol{\rho}}} = \frac{d\hat{\boldsymbol{\rho}}}{d\phi} \cdot \frac{d\phi}{dt} = \dot{\phi}\hat{\boldsymbol{\phi}}, \qquad \dot{\hat{\boldsymbol{\phi}}} = \frac{d\hat{\boldsymbol{\phi}}}{d\phi} \cdot \frac{d\phi}{dt} = -\dot{\phi}\hat{\boldsymbol{\rho}}, \qquad \dot{\hat{\boldsymbol{k}}} = \vec{0}$$

2.2. Coordenadas:

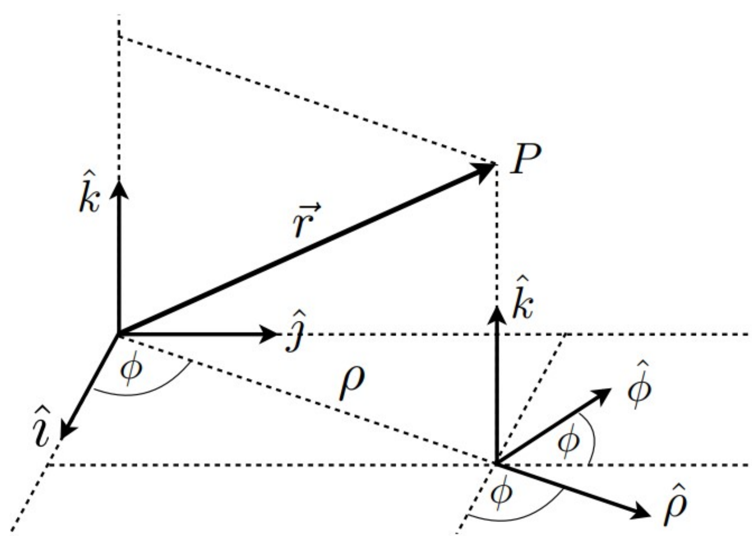
$$\rho \in [0, \infty), \qquad \phi \in [0, 2\pi), \qquad z \in (-\infty, \infty)$$

2.3. Vectores Cinemáticos:

Posición: $\vec{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\boldsymbol{k}}$

Velocidad: $\vec{v} = \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\boldsymbol{k}}$

Aceleración:
$$\begin{aligned}\vec{a} &= (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\boldsymbol{\rho}} + (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\hat{\boldsymbol{\phi}} + \ddot{z}\hat{\boldsymbol{k}} \\ &= (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{d}{dt}(\rho^2\dot{\phi})\hat{\boldsymbol{\phi}} + \ddot{z}\hat{\boldsymbol{k}}\end{aligned}$$



3. Coordenadas Esféricas

3.1. Vectores base $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$ y sus derivadas temporales:

$$\hat{\boldsymbol{r}} = \sin\theta\cos\phi\hat{\boldsymbol{x}} + \sin\theta\sin\phi\hat{\boldsymbol{y}} + \cos\theta\hat{\boldsymbol{k}}, \qquad \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\hat{\boldsymbol{x}} + \cos\theta\sin\phi\hat{\boldsymbol{y}} - \sin\theta\hat{\boldsymbol{k}}, \qquad \hat{\boldsymbol{\phi}} = -\sin\phi\hat{\boldsymbol{x}} + \cos\phi\hat{\boldsymbol{y}}$$

$$\dot{\hat{\boldsymbol{r}}} = \frac{\partial\hat{\boldsymbol{r}}}{\partial\theta}\cdot\frac{\partial\theta}{\partial t} + \frac{\partial\hat{\boldsymbol{r}}}{\partial\phi}\cdot\frac{\partial\phi}{\partial t} = \dot{\theta}\hat{\boldsymbol{\theta}} + \dot{\phi}\sin\theta\hat{\boldsymbol{\phi}}, \qquad \dot{\hat{\boldsymbol{\theta}}} = \frac{\partial\hat{\boldsymbol{\theta}}}{\partial\theta}\cdot\frac{\partial\theta}{\partial t} + \frac{\partial\hat{\boldsymbol{\theta}}}{\partial\phi}\cdot\frac{\partial\phi}{\partial t} = -\dot{\theta}\hat{\boldsymbol{r}} + \cos\theta\dot{\phi}\hat{\boldsymbol{\phi}}, \qquad \dot{\hat{\boldsymbol{\phi}}} = -(\sin\theta\hat{\boldsymbol{r}} + \cos\theta\hat{\boldsymbol{\theta}})\dot{\phi}$$

3.2. Coordenadas:

$$r \in [0, \infty], \qquad \theta \in [0, \pi], \qquad \phi \in [0, 2\pi)$$

3.3. Vectores Cinemáticos:

Posición: $\vec{r} = r\hat{\boldsymbol{r}}$

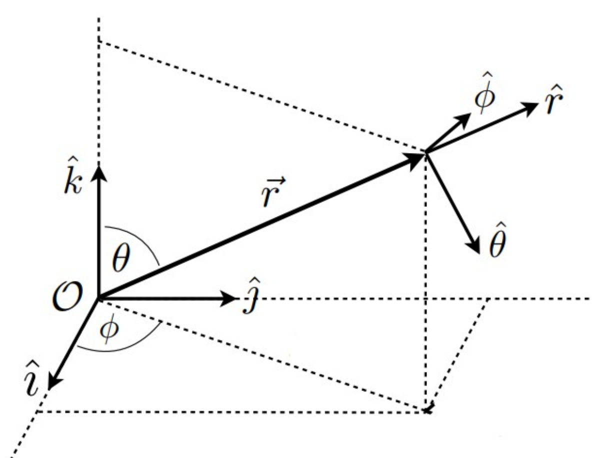
Velocidad: $\vec{v} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\phi}\sin\theta\hat{\boldsymbol{\phi}}$

Aceleración:

$$\begin{aligned}\vec{a} &= (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\hat{\boldsymbol{r}} \\ &+ (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\hat{\boldsymbol{\theta}} \\ &+ (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta)\hat{\boldsymbol{\phi}}\end{aligned}$$

La componente en ϕ de la aceleración también puede ser escrita como:

$$a_\phi = \frac{1}{r\sin\theta}\frac{d}{dt}(r^2\dot{\phi}\sin^2\theta)$$



4. Extras

4.1. Velocidad Angular: (Medida desde un origen O)

$$\vec{\omega} = \frac{\vec{r} \times \vec{v}}{\|\vec{r}\|^2}$$

4.2. Vectores tangente y normal: (Coordenadas Intrínsecas)

- Rapidez: (Notación)

$$\nu = \|\vec{v}\|$$

- Vectores Base:

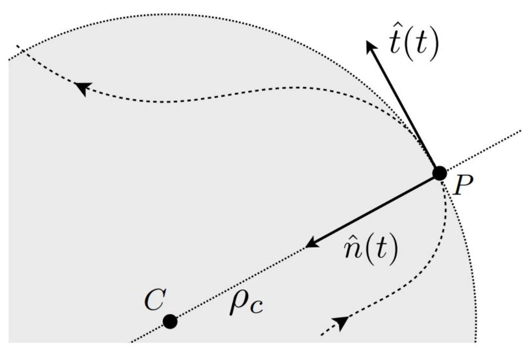
$$\hat{\boldsymbol{t}} = \frac{\vec{v}}{\nu}, \qquad \hat{\boldsymbol{n}} = \left\| \frac{d\hat{\boldsymbol{t}}}{dt} \right\|^{-1} \frac{d\hat{\boldsymbol{t}}}{dt}$$

- Radio de Curvatura:

$$\rho_c = \frac{\nu^3}{\|\vec{v} \times \vec{a}\|}$$

- Vectores Cinemáticos:

$$\vec{v} = \nu \hat{\boldsymbol{t}}, \qquad \vec{a} = \dot{\nu} \hat{\boldsymbol{t}} + \frac{\nu^2}{\rho_c} \hat{\boldsymbol{n}}$$



4.3. Cambios de variable:

- Cartesianas:
  - Desde variables cilíndricas: $(\rho, \phi, z) \longrightarrow (x, y, z)$

$$x = \rho \cos\phi, \qquad y = \rho \sin\phi, \qquad z = z$$

  - Desde variables esféricas: $(r, \theta, \phi) \longrightarrow (x, y, z)$

$$x = r \sin\theta \cos\phi, \qquad y = r \sin\theta \sin\phi, \qquad z = r \cos\theta$$

- Cilíndricas:
  - Desde variables cartesianas: $(x, y, z) \longrightarrow (\rho, \phi, z)$

$$\rho = \sqrt{x^2 + y^2}, \qquad \phi = \arctan\left(\frac{y}{x}\right), \qquad z = z$$

  - Desde variables esféricas: $(r, \theta, \phi) \longrightarrow (\rho, \phi, z)$

$$\rho = r \sin\theta, \qquad \phi = \phi, \qquad z = r \cos\theta$$

- Esféricas:
  - Desde variables cartesianas: $(x, y, z) \longrightarrow (r, \theta, \phi)$

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right), \qquad \phi = \arctan\left(\frac{y}{x}\right)$$

  - Desde variables cilíndricas: $(\rho, \phi, z) \longrightarrow (r, \theta, \phi)$

$$r = \sqrt{\rho^2 + z^2}, \qquad \theta = \arctan\left(\frac{\rho}{z}\right), \qquad \phi = \phi$$

4.4. Vectores en otras bases:

- Cartesianas:

  - Respecto a base cilíndrica: $\{\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{k}}\} \longrightarrow \{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{k}}\}$

$$\hat{\boldsymbol{x}} = \cos\phi\hat{\boldsymbol{\rho}} - \sin\phi\hat{\boldsymbol{\phi}}$$
$$\hat{\boldsymbol{y}} = \sin\phi\hat{\boldsymbol{\rho}} + \cos\phi\hat{\boldsymbol{\phi}}$$
$$\hat{\boldsymbol{k}} = \hat{\boldsymbol{k}}$$

  - Respecto a base esférica: $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\} \longrightarrow \{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{k}}\}$

$$\hat{\boldsymbol{x}} = \cos\phi\sin\theta\hat{\boldsymbol{r}} + \cos\phi\cos\theta\hat{\boldsymbol{\theta}} - \sin\phi\hat{\boldsymbol{\phi}}$$
$$\hat{\boldsymbol{y}} = \sin\phi\sin\theta\hat{\boldsymbol{r}} + \sin\phi\cos\theta\hat{\boldsymbol{\theta}} + \cos\phi\hat{\boldsymbol{\phi}}$$
$$\hat{\boldsymbol{k}} = \cos\theta\hat{\boldsymbol{r}} - \sin\theta\hat{\boldsymbol{\theta}}$$

- Cilíndricas:

  - Respecto a base cartesiana: $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{k}}\} \longrightarrow \{\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{k}}\}$

$$\hat{\boldsymbol{\rho}} = \cos\phi\hat{\boldsymbol{x}} + \sin\phi\hat{\boldsymbol{y}}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\boldsymbol{x}} + \cos\phi\hat{\boldsymbol{y}}$$
$$\hat{\boldsymbol{k}} = \hat{\boldsymbol{k}}$$

  - Respecto a base esférica: $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\} \longrightarrow \{\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{k}}\}$

$$\hat{\boldsymbol{\rho}} = \sin\theta\hat{\boldsymbol{r}} + \cos\theta\hat{\boldsymbol{\theta}}$$
$$\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$$
$$\hat{\boldsymbol{k}} = \cos\theta\hat{\boldsymbol{r}} - \sin\theta\hat{\boldsymbol{\theta}}$$

- Esféricas:

  - Respecto a base cartesiana: $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{k}}\} \longrightarrow \{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$

$$\hat{\boldsymbol{r}} = \sin\theta\cos\phi\hat{\boldsymbol{x}} + \sin\theta\sin\phi\hat{\boldsymbol{y}} + \cos\theta\hat{\boldsymbol{k}}$$
$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\hat{\boldsymbol{x}} + \cos\theta\sin\phi\hat{\boldsymbol{y}} - \sin\theta\hat{\boldsymbol{k}}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\boldsymbol{x}} + \cos\phi\hat{\boldsymbol{y}}$$

  - Respecto a base cilíndrica: $\{\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{k}}\} \longrightarrow \{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$

$$\hat{\boldsymbol{r}} = \sin\theta\hat{\boldsymbol{\rho}} + \cos\theta\hat{\boldsymbol{k}}$$
$$\hat{\boldsymbol{\theta}} = \cos\theta\hat{\boldsymbol{\rho}} - \sin\theta\hat{\boldsymbol{k}}$$
$$\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$$