P2 $U(x,t) = X(x)T(t)$

Reemplazando
$\Rightarrow$

$$\frac{\partial^2 U}{\partial t^2} = X(x)T''(t) \quad , \quad \frac{\partial^4 U}{\partial x^4} = X^{(4)}(x)T(t)$$

$$\Rightarrow \quad X(x)T''(t) + c^2 X^{(4)}(x)T(t) = 0$$

$$\left.\begin{aligned} X(0)T(t) &= X^{(2)}(0)T(t) = 0 \\ X(L)T(t) &= X^{(2)}(L)T(t) = 0 \end{aligned}\right\} \text{CB}$$

$$\left.\begin{aligned} X(x)\cdot T(0) &= f(x) \\ X(x)T'(0) &= g(x) \end{aligned}\right\} \text{CI}$$

a) Tomando la primera $\Rightarrow$ $\dfrac{X^{(4)}(x)}{X(x)} = -\dfrac{T''(t)}{c^2 T(t)} = -\alpha$

Usando solo info de X
$\Rightarrow$

$$X^{(4)}(x) = -\alpha X(x) \Rightarrow X^{(4)}(x) + \alpha X(x) = 0$$

CB
$$\Rightarrow X(0) = X^{(2)}(0) = 0$$
$$X(L) = X^{(2)}(L) = 0$$

b) Si $\alpha = 0 \Rightarrow X^{(4)}(x) = 0 \Rightarrow X(x) = A + Bx + Cx^2 + Dx^3$

Usando CB $\overset{X(0)=0}{\Rightarrow} A = 0 \quad \Rightarrow X(x) = Bx + Cx^2 + Dx^3$

$$\Rightarrow X^{(2)}(x) = 2C + 6Dx$$

$$X^{(2)}(0)=0 \Rightarrow C=0$$

$$\Rightarrow X(x)=Bx+Dx^3$$

$$\Rightarrow X^{(2)}(x)=6Dx$$

$$X(L)=0 \Rightarrow BL+DL^3=0 \quad BL=0 \Rightarrow B=0$$

$$X^{(2)}(L)=0 \Rightarrow 6DL=0 \Rightarrow D=0$$

$$\Rightarrow X(x)=0$$

c) $\int_0^L \alpha X(x)^2 dx + \int_0^L X^{(4)}(x) X(x)\, dx = 0$

IPP

$$\Rightarrow \int_0^L \alpha X(x)^2 dx + \underbrace{\cancel{X^{(3)}(x) X(x)\Big|_0^L}}_{X(0)=X(L)=0} - \int_0^L X^{(3)}(x) X'(x)\, dx = 0$$

$$\Rightarrow \alpha\int_0^L X^2(x) dx - \int_0^L X^{(3)}(x) X'(x)\, dx = 0$$

IPP

$$\Rightarrow \alpha\int_0^L X(x)^2 dx - \underbrace{\cancel{X^{(2)}(x) X'(x)\Big|_0^L}}_{X^{(2)}(0)=X^{(2)}(L)=0} + \int_0^L \left(X^{(2)}(x)\right)^2 dx = 0$$

$$\Rightarrow \boxed{\alpha\int_0^L X(x)^2 dx + \int_0^L \left(X^{(2)}(x)\right)^2 dx = 0}$$

d) Si $\alpha > 0 \Rightarrow X(x)=0$, pues sino

imposible $\int_0^L X(x)^2 dx = 0$

e)

$\lambda = -a^4 \Rightarrow X^{(4)}(x) - a^4 X(x) = 0$

$\Rightarrow X(x) = A\cosh(ax) + B\sinh(ax) + C\cos(ax) + D\sin(ax)$

$X(0) = 0 \Rightarrow A + C = 0$

$X^{(2)}(0) = 0 \Rightarrow a^2 A - a^2 C = 0 \Rightarrow A - C = 0$

$\Rightarrow A = C = 0$

$\Rightarrow X(x) = B\sinh(ax) + D\sin(ax)$

$X(L) = 0 \Rightarrow B\sinh(aL) + D\sin(aL) = 0$

$X^{(2)}(L) = 0 \Rightarrow a^2(B\sinh(aL) - D\sin(aL)) = 0$

$\Rightarrow B\sinh(aL) = 0 \Rightarrow B = 0$, pues $\sinh(aL) \neq 0$ si $aL \neq 0$

$\Rightarrow D\sin(aL) = 0 \Rightarrow aL = n\pi$

$\Rightarrow a = \frac{n\pi}{L}$, $n \in \mathbb{N}^*$

$\Rightarrow X_n(x) = D_n \sin\left(\frac{n\pi}{L}x\right)$

f) $\frac{T_n''(t)}{-c^2 T_n(t)} = -a^4 = -\frac{n^4\pi^4}{L^4} \Rightarrow T_n''(t) = \frac{n^4\pi^4}{L^4} c^2 T_n(t)$

$\Rightarrow T_n(t) = A_n\cos\left(\frac{n^2\pi^2 c}{L^2}t\right) + B_n\sin\left(\frac{n^2\pi^2 c}{L^2}t\right)$

$U(x,t) = \sum_{n=1}^{\infty} U_n(x)$

Principio Superposición

$\Rightarrow U_n(t,x) = \left[A_n\cos\left(\frac{n^2\pi^2 c}{L^2}t\right) + B_n\sin\left(\frac{n^2\pi^2 c}{L^2}t\right)\right]\sin\left(\frac{n\pi}{L}x\right)$

9) $f(x) = \sum_{N=1}^{\infty} a_N \sin(\frac{N\pi}{L}x)$, con $a_N = \frac{2}{L}\int_0 f(x)\sin(\frac{N\pi}{L}x)dx$

en este caso nos dan que $a_4 = 5$ y el resto es 0

Nuestro $U(t,x)$ en $t=0$ es igual

$$\Rightarrow U(0,x) = \sum_{N=1}^{\infty} A_N \sin(\frac{N\pi}{L}x) = 5\sin(\frac{4\pi}{L}x)$$

$$\Rightarrow A_N = 0 \quad \forall N \in \mathbb{N},\ N \neq 4, \quad A_4 = 5$$

$$U_t(0,x) = \sum_{N=1}^{\infty} \frac{B_N \cdot N\pi}{L} \sin(\frac{N\pi x}{L}) = g(x) = -7\sin(\frac{4\pi x}{L})$$

$$\Rightarrow B_N = 0 \quad \forall N \neq 4, \quad \frac{B_4 \cdot 4\pi}{L} = -7$$

$$\Rightarrow B_4 = \frac{-7L}{4\pi}$$

$$\Rightarrow U(t,x) = 5\cos(\frac{16\pi^2 c}{L}t)\sin(\frac{4\pi}{L}x) +$$
$$\frac{-7L}{4\pi}\sin(\frac{16\pi^2 c}{L}t)\sin(\frac{4\pi}{L}x)$$

$\boxed{P1}$ Recomiendo primero $\underline{P2}$

$$U(t,x) = T(t)\,X(x)$$

$$T''(t)X(x) + a\,T'(t)X(x) = T(t)\,X''(x)$$

$$X(0)T(t) = X(\pi)T(t) = 0 \Rightarrow X(0) = X(\pi) = 0$$

$\Rightarrow$

$$\frac{X''(x)}{X(x)} = \frac{T''(t) + a\,T'(t)}{T(t)} = \alpha$$

$$\Rightarrow X''(x) - \alpha X(x) = 0$$

$\boxed{\text{Si } \alpha = 0}$ $\Rightarrow X''(x) = 0 \Rightarrow X(x) = Ax + B$

Si $X(0) = 0 \Rightarrow B = 0 \Rightarrow X(x) = Ax$

Si $X(\pi) = 0 \Rightarrow A\pi = 0 \Rightarrow A = 0 \Rightarrow X \equiv 0$

Caso Trivial

$\boxed{\text{Si } \alpha > 0}$ $\Rightarrow X(x) = A\cosh(\sqrt{\alpha}\,x) + B\sinh(\sqrt{\alpha}\,x)$

$X(0) = 0 \Rightarrow A = 0 \Rightarrow X(x) = B\sinh(\sqrt{\alpha}x)$

$X(\pi) = 0 \Rightarrow B\sinh(\sqrt{\alpha}\,\pi) = 0 \Rightarrow B = 0$

$\Rightarrow$ Caso Trivial

$\Rightarrow \alpha < 0$, Tomemos $\alpha = -b^2$, $a > 0$

$\Rightarrow X''(x) + b^2 X(x) = 0$

$\Rightarrow X(x) = A\cos(bx) + B\sin(bx)$

$X(0) = 0 \Rightarrow A = 0 \Rightarrow X(x) = B\sin(bx)$

$X(\pi) = 0 \Rightarrow \cancel{X(\pi)}\ B\sin(b\pi) = 0 \Rightarrow b\pi = k\pi$

Con $k \in \mathbb{N}^* \Rightarrow \boxed{X_k(x) = B_k \sin(kx)}$

Ahora $T''(t) + aT'(t) = -k^2 T(t)$

$\Rightarrow T''(t) + aT'(t) + k^2 T(t) = 0$

$\Rightarrow \lambda^2 + a\lambda + k^2 = 0 \Rightarrow \lambda = \dfrac{-a \pm \sqrt{a^2 - 4k^2}}{2}$

$= \dfrac{-a \pm \sqrt{4k^2 - a^2}\, i}{2}$

$\Rightarrow T_k(t) = \left[C_k \cos\left(\frac{\sqrt{4k^2 - a^2}}{2} t\right) + D_k \sin\left(\frac{\sqrt{4k^2 - a^2}}{2} t\right)\right] e^{-\frac{a}{2} t}$

$\Rightarrow U_k(t,x) = \left[C_k \cos\left(\frac{\sqrt{4k^2 - a^2}}{2} t\right) + D_k \sin\left(\frac{\sqrt{4k^2 - a^2}}{2} t\right)\right] e^{-\frac{a}{2} t}$

$\cdot \sin(kx)$

Por superposición

$$\lambda_k = \frac{\sqrt{4k^2 - a^2}}{2}$$

$$U(t,x) = \sum_{k=1}^{\infty} \left[C_k \cos(\lambda_k t) + D_k \sin(\lambda_k t) \right] e^{-\frac{a}{2}t} \cdot \sin(kx)$$

Como

$$f(x) = \sum_{k=1}^{\infty} a_k \sin(kx) \quad , \text{ con } \boxed{a_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(kx) dx}$$

$$\Rightarrow U(0,x) = f(x) \Rightarrow \boxed{C_k = a_k}$$

$$Y \ U_t(0,x) = \sum_{k=1}^{\infty} \left[\underset{\substack{\downarrow \\ \text{es } a_k}}{C_k} \left(-\frac{a}{2}\right) + \lambda_k D_k \right] \sin(kx) = 0 = \sum_{k=1}^{\infty} 0 \sin(kx)$$

$$\Rightarrow -\frac{a_k a}{2} + \lambda_k D_k = 0$$

$$\Rightarrow \boxed{D_k = \frac{a_k \cdot a}{2 \lambda_k}}$$

Con lo que se tiene todo $\blacksquare$