P1 a)

a) Por def:

$$\frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-i\nu x} e^{-2x}\,dx + \int_{-\infty}^{0} e^{-i\nu x} e^{5x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_0^{\infty} e^{-x(2+i\nu)}\,dx + \int_{-\infty}^{0} e^{x(5-i\nu)}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-x(2+i\nu)}}{-(2+i\nu)}\Bigg|_0^{\infty} + \frac{e^{x(5-i\nu)}}{5-i\nu}\Bigg|_{-\infty}^{0}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{2+i\nu} + \frac{1}{5-i\nu}\right] = \boxed{\frac{7}{\sqrt{2\pi}(\nu^2+3i\nu+10)}}$$

Con Propiedades

a) $f(x) = e^{-2x}\mathbb{1}_{x\geq 0} + e^{5x}\mathbb{1}_{x<0} \Rightarrow f(x) = f_1(x) + f_2(x)$

Notar que $f_1(x) = g(2x)$, $f_2(x) = g(-5x)$, con $g(x) = \begin{cases} e^{-x} & x\geq 0 \\ 0 & x<0 \end{cases}$

$$\Rightarrow \widehat{f_1(x)}(\nu) = \widehat{g(2x)}(\nu) = \frac{1}{2}\widehat{g(x)}\left(\frac{\nu}{2}\right) = \frac{1}{2+i\nu}\cdot\frac{1}{\sqrt{2\pi}}$$

$$\widehat{f_2(x)}(\nu) = \widehat{g(-5x)}(\nu) = \frac{1}{5}\widehat{g(x)}\left(\frac{\nu}{-5}\right) = \frac{1}{5-i\nu}\cdot\frac{1}{\sqrt{2\pi}}$$

$$\boxed{\widehat{f}(x)(\nu) = \frac{1}{\sqrt{2\pi}}\frac{7}{\nu^2+3i\nu+10}}$$

b) $$f(x) = \frac{1}{x^2+6x+13} = \frac{1}{x^2+6x+9+4} = \frac{1}{(x+3)^2+2^2} = g_2(x+3)$$

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Con $$g_a(x) = \frac{1}{x^2+a^2} \Rightarrow \widehat{g_2(x)}(\nu) = \underbrace{\cancel{\sqrt{\frac{2}{\pi}}}}_{\sqrt{\frac{\pi}{2}}} \cdot \frac{1}{2} \cdot e^{-2|\nu|}$$

$$\hat{f}(x)(\nu) = \widehat{g_2(x+3)}(\nu) = e^{3i\nu}\, \widehat{g_2(x)}(\nu) = \frac{1}{2}\underbrace{\cancel{\sqrt{\frac{2}{\pi}}}}_{\sqrt{\frac{\pi}{2}}}\, e^{3i\nu} e^{-2|\nu|}$$

$$\boxed{\hat{f}(x)(\nu) = \frac{1}{2}\sqrt{\frac{\pi}{2}}\, e^{i\nu 3} \cdot e^{-2|\nu|}}$$

c)

$y(x) = x e^{-3x} \mathbb{1}_{\{x>0\}}$, Notar que $\widehat{e^{-ax} \mathbb{1}_{x\geq 0}}(\nu) = \frac{1}{\sqrt{2\pi}} \frac{1}{a+i\nu}$
$a>0$

$$y'(x) = -3y(x) + e^{-3x} \mathbb{1}_{\{x\geq 0\}}$$

$$\Rightarrow \widehat{y'(x)}(\nu) = -3 \widehat{y(x)}(\nu) + \frac{1}{\sqrt{2\pi}(3+i\nu)}$$

$$\Rightarrow (3+i\nu) \widehat{y(x)}(\nu) = \frac{1}{\sqrt{2\pi}(3+i\nu)}$$

$$\Rightarrow \widehat{y(x)}(\nu) = \frac{1}{\sqrt{2\pi}(3+i\nu)^2}$$

P2 a) $\hat{f(x)}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{i\lambda x} dx = \frac{1}{\sqrt{2\pi}} \frac{e^{i\lambda x}}{i\lambda} \Big|_{-1}^{1} = \frac{1}{\sqrt{2\pi}} \frac{e^{i\lambda} - e^{-i\lambda}}{i\lambda}$

$= \sqrt{\frac{2}{\pi}} \frac{\sin(\lambda)}{\lambda}$

$a = \frac{1}{4}$

$\sqrt{2} e^{-\lambda^2} = e^{-\frac{x^2}{4}}$

b) $\hat{g}(\lambda) = \sqrt{\frac{2}{\pi}} \frac{\sin(\lambda)}{\lambda} \cdot e^{-\lambda^2} = \hat{f(x)} \cdot (\widehat{e^{-\lambda^2}})$

$\widehat{g(x)}(\lambda) = \hat{f(x)} \cdot \underbrace{\widehat{\frac{e^{-\frac{x^2}{4}}}{\sqrt{2}}}}_{h(x)} = \widehat{\frac{f * (h)(x)}{\sqrt{2\pi}}}(\lambda)$

$\Rightarrow g(x) = \frac{f * h\,(x)}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 1_{[-1,1]}(y)\, e^{-\frac{(y-x)^2}{4}} dy$

$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-\frac{(y-x)^2}{4}} dy$

$\boxed{P3}$ $y'' - y = e^{-ax} \mathbb{1}_{x\geq 0}$ $\quad / \widehat{(\ )}$

$$\widehat{y''(x)}(\lambda) - \widehat{y(x)}(\lambda) = \frac{1}{\sqrt{2\pi}} \frac{1}{a+i\lambda}$$

$$-(\lambda^2+1)\hat{y}(\lambda) = \frac{1}{\sqrt{2\pi}} \frac{1}{a+i\lambda}$$

$$\Rightarrow \hat{y}(\lambda) = \frac{-1}{\sqrt{2\pi}} \frac{1}{(\lambda^2+1)} \frac{1}{(a+i\lambda)} \longrightarrow \boxed{-\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{(\lambda^2+1)(a+i\lambda)}}$$

$$= -\frac{1}{2\pi} \sqrt{2\pi} \left(\frac{1}{\lambda^2+1}\right)\left(\frac{1}{a+i\lambda}\right) \qquad \frac{1}{\lambda^2+1} = \widehat{\sqrt{\tfrac{\pi}{2}}\, e^{-|x|}}(\lambda)$$

$$= -\frac{1}{2\pi} \widehat{h(x) * f(x)}(\lambda)$$

$h(x) = \sqrt{\frac{\pi}{2}} e^{-|x|}$

$f(x) = e^{-ax} \mathbb{1}_{x\geq 0}$

$$\Rightarrow y(x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ay} \mathbb{1}_{y\geq 0} \sqrt{\frac{\pi}{2}} e^{-|y-x|}\, dy$$

$$\boxed{y(x) = -\frac{1}{2\sqrt{\pi}}\int_0^{\infty} e^{-ay} e^{-|y-x|}\, dy}$$