$\boxed{P1}$ Si $a = \frac{1}{2}$

$$\oint_{|z|=a} \frac{e^{\pi z}}{z(z^2+1)}\,dz \quad \text{Solo 1 polo } z=0$$

$$\lim_{z\to 0} z f(z) = \lim_{z\to 0} \frac{e^{\pi z}}{z^2+1} = \boxed{1} \Rightarrow \boxed{\oint f(z)dz = 2\pi i}$$

Si $a = 2 \Rightarrow 3$ polos $z=0$ y $z=\pm i$

$$\lim_{z\to i} (z-i) f(z) = \lim_{z\to i} \frac{e^{z\pi}}{z(z+i)} = \frac{(-1)}{i(2i)} = \boxed{\frac{1}{2}}$$

$$\lim_{z\to -i} (z+i) f(z) = \lim_{z\to -i} \frac{e^{z\pi}}{z(z-i)} = \frac{(-1)}{(-i)(-2i)} = \boxed{\frac{1}{2}}$$

$$\Rightarrow \operatorname{Res}(f, P_1, P_2, P_3) = 2 \Rightarrow \boxed{\oint_{|z|=a} f(z)dz = 4\pi i}$$

$\boxed{P2}$ $\displaystyle\oint_{|z|=4} \frac{1}{z^2 \sinh(z)}\,dz$

$\sinh(z) = -i\sin(iz)$

$\Rightarrow$ Ceros son $in\pi$

Solo importa $-\pi i, 0, \pi i$

pues el resto $|z| > 4$

$$\lim_{z\to 0} z^3 f(z) = \lim_{z\to 0} \frac{z}{\sinh(z)} = \lim_{z\to 0} \frac{z}{(-i)\sin(iz)}$$

$$\overset{L'H}{=} \lim_{z\to 0} \frac{1}{(-i)(i)\cos(iz)} = \underline{1} \Rightarrow \text{orden } 3$$

$$z^3 f(z) = \frac{z}{(-i)\sin(iz)} = \frac{z}{\sinh(z)}$$

$$\frac{d}{dz}\left(z^3 f(z)\right) = \frac{\sinh(z) - z\cosh(z)}{\sinh(z)^2}$$

$$\frac{d^2}{dz^2}\left(z^3 f(z)\right) = \frac{\left[\cancel{\cosh(z)} - \cancel{\cosh(z)} - z\sinh(z)\right]\sinh(z)^2 - 2\sinh(z)\cosh(z)\left[\sinh(z) - z\cosh(z)\right]}{\sinh(z)^4}$$

$$\frac{d^3}{dz^2}\left(z^3 f(z)\right) = \frac{-z}{\sinh(z)} + \frac{-2\cosh(z)\sinh(z) + 2z\cosh(z)^2}{\sinh^3(z)}$$

$$\overset{L'H}{=} \frac{-1}{\cosh(z)} + \frac{(-2)\sinh^2(z) - \cancel{2\cosh^2(z)} + \cancel{2\cosh^2(z)} + 4z\cosh(z)}{3\sinh^2(z)\cosh(z)}$$

$$= \frac{-1}{\cosh(z)} - \frac{2}{3\cosh(z)} + \frac{4}{3}\frac{z}{\sinh(z)}$$

$$\overset{z \to 0}{=} -1 - \frac{2}{3} + \frac{4}{3} = \boxed{-1/3}$$

$$\Longrightarrow \operatorname{Res}(f, 0) = \boxed{-1/3}$$

$$\text{Residuo}(f, 0) = \frac{1}{2!} \cdot 1/3 = \boxed{\frac{1}{6}}$$

$i\pi$ y $-i\pi$ polos simples

$$\lim_{z \to i\pi} (z - i\pi) f(z) = \lim_{z \to i\pi} \frac{(z - i\pi)}{z^2 \sinh(z)} \overset{L'H}{=} \lim_{z \to i\pi} \frac{1}{2z\sinh(z) + z^2\cosh(z)}$$

$$= \frac{1}{0 + \pi^2} = \boxed{\frac{1}{\pi^2}}$$

$\cosh(i\pi) = \cos(\pi) = -1$

$$\lim_{z \to (-i\pi)} \left((z + i\pi) f(z) \right) = \lim_{z \to -i\pi} \frac{(z+i\pi)}{z^2 \sinh(z)}$$

$$\overset{L'H}{=} \lim_{z \to -i\pi} \frac{1}{2z\sinh(z) + z^2\cosh(z)} = \frac{1}{0 + (-\pi^2)(1)}$$

$$= \boxed{\frac{1}{\pi^2}}$$

$$\Rightarrow \oint_{|z|=4} f(z)\, dz = \boxed{2\pi i \left(\frac{2}{\pi^2} + \frac{1}{6} \right)}$$

$\boxed{P3}$ Por el AUX la

Serie de $|x|$ de $[-1,1]$ es

la d) $\boxed{P_1}$ $\Rightarrow$ $\text{Serie} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-2)}{n^2\pi^2}\left[1-(-1)^n\right]\cos(n\pi x)$

Notando que $1+(-1)^n = \begin{cases} 0 & \text{si } n \text{ par} \\ 2 & \text{si } n \text{ impar} \end{cases}$

$$\text{Serie} = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-2)(2)}{(2k-1)^2\pi^2}\cos((2k-1)\pi x)$$

$\rightarrow$ Se fueron los pares

Se puede evaluar en muchos x

Tomemos $\boxed{x=0}$ $\Rightarrow |x|=0$

$$0 = \frac{1}{2} + \left(\frac{-4}{\pi^2}\right)\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$$

$$\Rightarrow \boxed{\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}}$$

$\boxed{P4}$ $f(x) = x - [x]$ Con $x \in [0,1]$

$\Downarrow$ $2L$ $\Rightarrow L = 1/2$

Notamos que $[x] = 0$ para $x \in [0,1)$

$$a_0 = \frac{1}{1/2}\int_{-1/2}^{1/2} f(x)\,dx \overset{\text{Paridad}}{=} 2\int_0^1 f(x)\,dx$$

$$= 2\int_0^1 x - [x]\,dx = 2\int_0^1 x\,dx = x^2\Big|_0^1 = \boxed{1}$$

$$\Rightarrow \boxed{\frac{a_0}{2} = \frac{1}{2}}$$

$$a_N = \frac{1}{1/2}\int_{-1/2}^{1/2} f(x)\cos\left(\frac{N\pi x}{1/2}\right)dx = 2\int_0^1 (x - [x])\cos(2N\pi x)\,dx$$

$$= 2\int_0^1 x\cos(2N\pi x)\,dx = 2\left[\cancelto{0}{\frac{x\sin(2N\pi x)}{2N\pi}} - \int_0^1 \frac{\sin(2N\pi x)}{2N\pi}\,dx\right]$$

$$= 2\left[\frac{\cos(2N\pi x)}{(2N\pi)^2}\Big|_0^1\right] = 2\left[\frac{1-1}{(2N\pi)^2}\right] = 0$$

$$b_n = 2\int_{-1/2}^{1/2} f(x)\, \sin(2n\pi x)\, dx$$

$$= 2\int_0^1 (x-[x])\, \sin(2n\pi x)\, dx$$

$$= 2\int_0^1 x \sin(2n\pi x)\, dx$$

$$= 2\left[-x \frac{\cos(2n\pi x)}{2n\pi}\Big|_0^1 + \int_0^1 \cancel{\frac{\cos(2n\pi x)}{2n\pi}}^{\,0} dx \right]$$

$$= \frac{(-2)(1)}{2n\pi} = \boxed{\frac{-1}{n\pi}}$$

$$\Rightarrow \boxed{\text{Seri} = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin(2n\pi x)}$$