P1 a) $\oint_{D(0,3)} \frac{e^{2z}}{(z+1)^4}$

Usando $f(z) = e^{2z}$

$$f^{(3)}(z_0) = \frac{3!}{2\pi i} \oint_{D(0,3)} \frac{f(z)}{(z-z_0)^{3+1}} dz$$

$$\Rightarrow 2^3 e^{-2} = \frac{3!}{2\pi i} \oint_{D(0,3)} \frac{f(z)}{(z+1)^4} dz$$

$$\Rightarrow \boxed{\oint_{D(0,3)} \frac{e^{2z}}{(z+1)^4} dz = \frac{16\pi i}{3!\, e^2}} = \frac{8\pi i}{3e^2}$$

b) $\oint_{D(0,3)} \frac{\text{Cosh}(z^2)}{z\,(z^2+4)}\,dz \Rightarrow$ Polos $= \{-2i, 0, 2i\}$

Polo $(-2i)$

$$\lim_{z\to(-2i)} (z+2i)\,f(z) = \lim_{z\to -2i} \frac{\text{Cosh}(z^2)}{z\,(z-2i)}$$

$$= \frac{\text{Cosh}(-4)}{(-2i)(-4i)} = \boxed{\frac{-\text{Cosh}(4)}{8}}$$

Como order $1 \Rightarrow \text{Res}(f, -2i) = \frac{-\text{Cosh}(4)}{8}$

Polo $(2i)$

$$\lim_{z\to 2i} (z-2i)\,f(z) = \lim_{z\to 2i} \frac{\text{Cosh}(z^2)}{z\,(z+2i)} = \frac{\text{Cosh}(-4)}{(2i)\,(4i)}$$

$$= \boxed{\frac{-\text{Cosh}(4)}{8}}$$

Polo 0

$$\lim_{z\to 0} z\,f(z) = \lim_{z\to 0} \frac{\text{Cosh}(z^2)}{z^2+4} = \boxed{\frac{1}{4}}$$

Como Todos cumplen que $|z| \leq 3$

entonces pertenecen al disco

$$\Rightarrow \oint_{D(0,3)} \frac{\cosh(z^2)}{z(z^2+4)} dz = 2\pi i \left(\frac{1}{4} - \frac{\cosh(4)}{8} - \frac{\cosh(4)}{8} \right)$$

$$= \boxed{\frac{\pi i}{2} \left(1 - \cosh(4) \right)}$$

P2

$$\int_0^{2\pi} \frac{d\theta}{5-3\sin(\theta)} = \oint_{D(0,1)} \frac{\frac{dz}{iz}}{5-\frac{3}{2i}(z-z^{-1})}$$

$$= \oint_{D(0,1)} \frac{dz}{5iz - \frac{3i}{2i}(z^2-1)}$$

$$= \oint_{D(0,1)} \frac{2dz}{10iz - 3z^2 + 3}$$

$$z_{1,2} = \frac{-10i \pm \overset{8i}{\sqrt{-100+36}}}{-6} = 3i \ , \ \frac{1}{3}i$$

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{5-3\sin\theta} = \oint_{D(0,1)} \frac{2dz}{(-3)(z-3i)(z-\frac{1}{3}i)}$$

Solo importa $\frac{1}{3}i$ y por lo tanto

$$\lim_{z\to\frac{1}{3}i} \left(z-\frac{1}{3}i\right) f(z) = \lim_{z\to\frac{1}{3}i} \frac{2}{(-3)(z-3i)} = \left(-\frac{2}{3}\right)\frac{1}{\left(\frac{1}{3}i-3i\right)}$$

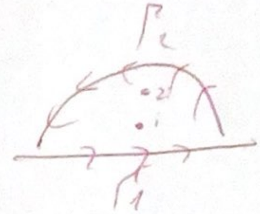
$$= \frac{-2}{1i-9i} = \frac{2}{8i} = \boxed{-\frac{i}{4}} \neq 0$$

$\Rightarrow$ Polo simple y Residuo $\boxed{-\frac{i}{4}}$

$$\Rightarrow \int_0^{2\pi} \frac{1}{5-3\operatorname{sen}\theta}\,d\theta = \boxed{\frac{\pi}{2}}$$

$\boxed{P3}$ $$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}\,dx$$

Considerar $f(z) = \dfrac{1}{(z^2+1)(z^2+4)}$ y



$$\lim_{z\to i} (z-i)f(z) = \lim_{z\to i} \frac{1}{(z+i)(z^2+4)} = \frac{1}{(2i)(3)} = \frac{1}{6i} \neq 0$$

Polo simple

$$\lim_{z\to 2i} (z-2i)f(z) = \lim_{z\to 2i} \frac{1}{(z^2+1)(z+2i)} = \frac{1}{(-3)(4i)} = -\frac{1}{12i} \neq 0$$

Polo Simple

$$\Rightarrow \oint_{\Gamma_1\cup\Gamma_2} f(z) = 2\pi i\left(\frac{1}{6i} - \frac{1}{12i}\right) = \boxed{\frac{\pi}{6}}$$

$$\left|\int_{\Gamma_2} f(z)dz\right| \leq \int_0^{\pi} \left|\frac{Ri e^{i\theta}\,d\theta}{(R^2e^{2i\theta}+1)(R^2e^{2i\theta}+4)}\right| \leq \int_0^{\pi} \frac{R\,d\theta}{|R^2e^{2i\theta}+1||R^2e^{2i\theta}+4|}$$

$$\left|\int_{\Gamma_2} f(z)\,dz\right| \leq \int_0^{\pi} \frac{R\,d\theta}{(R^2-1)(R^2-4)} = \frac{\pi R}{(R^2-1)(R^2-4)} \xrightarrow{R\to\infty} 0$$

$$\Rightarrow \int_{\Gamma_2} f(z)\,dz \longrightarrow 0$$

$$\int_{\Gamma_1} f(z)\,dz = \int_{-R}^{R} \frac{1}{(x^2+1)(x^2+4)}\,dx \longrightarrow \int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}\,dx$$

Como $\displaystyle\oint_{\Gamma_1\cup\Gamma_2} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{6}}$$

P4

$$\oint_{|z+i|=\frac{1}{2}} \frac{e^{iz}}{(z^2+1)^2}\,dz \longrightarrow \underbrace{(z+i)}_{\text{Solo ese importa}}(z-i)$$

$$\lim_{z\to -i} (z+i)^2 f(z) = \lim_{z\to -i} \frac{e^{iz}}{(z-i)^2} = \frac{e}{-1} = -e \neq 0$$

Polo de orden 2

$$\operatorname{Res}(f,-i) = \lim_{z\to -i} \frac{1}{1!} \frac{d}{dz}\left(\frac{e^{iz}}{(z-i)^2}\right)$$

$$= \lim_{z\to -i} \frac{(z-i)^2 e^{iz}\, i - 2(z-i)e^{iz}}{(z-i)^4}$$

$$= \lim_{z\to -i} \frac{e^{iz}\,((z-i)i - 2)}{(z-i)^3}$$

$$= \frac{e}{8i}(0) = 0$$

$$\Rightarrow \boxed{\oint_{|z+i|=\frac{1}{2}} \frac{e^{iz}}{(z^2+1)^2}\,dz = 0}$$

Risposta

$$\int_0^{2\pi} \frac{\cos\theta}{13+12\cos\theta}\,d\theta = \oint_{|z|=1} \frac{\frac{z^2+1}{2z}\,\frac{dz}{iz}}{13+\frac{12(z^2+1)}{2z}}$$

$$= \oint_{|z|=1} \frac{(z^2+1)\,dz}{zi(26z+12z^2+12)}$$

$$= \oint_{|z|=1} \frac{(z^2+1)\,dz}{2iz(13z+6z^2+6)}$$

$$z_{1,2} = \frac{-13 \pm \sqrt{169-144}}{12} = -\frac{3}{2},\ -\frac{2}{3}$$

Solo importa esa

$$= \oint_{|z|=1} \frac{(z^2+1)\,dz}{12iz\left(z+\frac{3}{2}\right)\left(z+\frac{2}{3}\right)}$$

$$\lim_{z\to 0} zf(z) = \frac{1}{12i\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)} = \boxed{\frac{1}{12i}}$$

$$\lim_{z\to -\frac{2}{3}} \left(z+\frac{2}{3}\right)f(z) = \frac{\frac{4}{9}+1}{(12i)\left(-\frac{2}{3}\right)\left(\frac{3}{2}-\frac{2}{3}\right)} = \frac{\frac{13}{9}}{i\left(-\frac{2}{3}\right)\underbrace{(18-8)}_{10}} = \boxed{\frac{-13}{60i}}$$

$$\Rightarrow \int_0^{2\pi} \frac{\cos\theta}{13+12\cos\theta}\,d\theta = 2\pi\cancel{i}\left(\frac{1}{12}-\frac{13}{60}\right) = -\pi\left(\frac{16}{60}\right) = \boxed{-\frac{4\pi}{15}}$$