P1 a) $p^k(z) = \exp(k\log(z)) = \exp(k[\ln(R) + i\theta])$

$= \exp(k\ln(R)) \cdot \exp(ik\theta)$

$= \exp(\ln(R^k)) \cdot e^{ik\theta}$

$= R^k e^{ik\theta}$

$= (Re^{i\theta})^k$

$= z^k$ ✓

$\overline{\overline{p^\lambda(\bar z)}} = \overline{\exp(\lambda\log(\bar z))} = \exp(\overline{\lambda\log(\bar z)}) = \exp(\bar\lambda\,\overline{\log(\bar z)})$

$= \exp(\bar\lambda\log(\bar{\bar z})) = \exp(\bar\lambda\log(z))$

$= p^{\bar\lambda}(z)$

* uso $\overline{a\cdot b} = \bar a \cdot \bar b$, $\overline{\exp(z)} = \exp(\bar z)$ y $\overline{\log(z)} = \log(\bar z)$

$\overline{\exp(z)} = \overline{\exp(x)\cdot e^{iy}} = \exp(x)\cdot e^{-iy} = \exp(x-iy) = \exp(\bar z)$

$\overline{\log(z)} = \overline{\ln(r) + i\theta} = \ln(r) - i\theta = \log(re^{-i\theta}) = \log(\bar z)$

b) $p^{\lambda+\mu}(z) = \exp((\lambda+\mu)\log(z)) = \exp(\lambda\log(z))\cdot\exp(\mu\log(z))$

$= p^\lambda(z)\, p^\mu(z)$

Notar que solo $\log(z)$ presenta restricción

$\Rightarrow$ es holomorfa en $\mathbb{C} - \{\mathbb{R}_-\}$

$$(p^\lambda(z))' = (\exp(\lambda \log(z)))' = \exp(\lambda \log(z))(\lambda \log(z))'$$

$$= p^\lambda(z) \cdot \lambda \cdot z^{-1}$$

$$\overset{a)}{=} p^\lambda(z)\, p^{-1}(z) \cdot \lambda$$

$$\overset{b)}{=} p^{\lambda-1}(z)\lambda$$

c) Entonces definimos

$$z^\lambda = \exp(\lambda \log(z)) \quad (1)$$

Probemos quien es $t^{\alpha+i\beta}$, para $t > 0$ real

Usando (1) $\Rightarrow$ $t^{\alpha+i\beta} = \exp((\alpha+i\beta)\log(t))$

$$= e^{\alpha \log(t)} \cdot e^{i\beta \log(t)}$$

$$= t^\alpha(\cos(\beta \log(t)) + i \sin(\beta \log(t)))$$

Finalmente $i^i = \exp(i \log(i))$ Usando $\log(i) = 0 + i\frac{\pi}{2}$

$$= \boxed{\exp(-\tfrac{\pi}{2})}$$

P2 a) Cambio a $\sum_{n=0}^{\infty} 4^n (z+2)^{2^n}$

$$a_k = \begin{cases} k^2 & k = 2^n \\ 0 & k \neq 2^n \end{cases}$$

$$\Rightarrow \limsup_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \sqrt[k]{k^2} = 1 \Rightarrow \boxed{R=1}$$

b) $\sum_{n\geq 0} (8+(-1)^n)^n z^n$, $a_n = \begin{cases} 9^n & n \text{ par} \\ 7^n & n \text{ impar} \end{cases}$

$$\Rightarrow \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{9^n} = 9 \Rightarrow \boxed{R = 1/9}$$

P3 $f(z) = \frac{z}{(1-z)^2}$, $\frac{1}{(1-z)} = \sum_{k=0}^{\infty} z^k$, derivando

$$\frac{-1}{(1-z)^2} = \sum_{k=1}^{\infty} k \cdot z^{k-1} \Rightarrow \boxed{\frac{z}{1-z^2} = \sum_{k=1}^{\infty} k\, z^k}$$

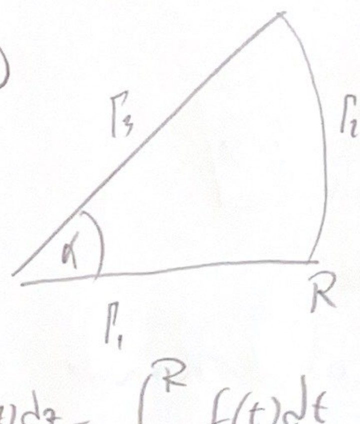
b) $g(z) = \frac{z^2}{(1+z)^2}$ / $\frac{1}{\underset{\underset{-(-z)}{\downarrow}}{1+z}} = \sum_{k=0}^{\infty} (-z)^k = \sum_{k=0}^{\infty} (-1)^k z^k$

$$\left(\frac{1}{1+z}\right)' = \frac{-1}{(1+z)^2} = \sum_{k=1}^{\infty} (-1)^{k} k z^{k-1}$$

$$\Rightarrow g(z) = \frac{z^2}{(1+z)^2} = \sum_{k=1}^{\infty} (-1)^{k+1} k z^{k+1}$$

$$= \boxed{\sum_{k=2}^{\infty} (-1)^k (k-1) z^k}$$

P4 a)



$\gamma_1(t) = t$, $t \in [0, R]$

$\gamma_2(t) = e^{it}$, $t \in [0, \alpha]$

$\gamma_3(t) = te^{i\alpha}$, $t \in [R, 0]$

$$\int_{\Gamma_1} f(z)dz = \int_0^R f(t)dt \longrightarrow \int_0^\infty f(t)dt$$

$$\int_{\Gamma_2} f(z)dz = \int_0^\alpha f(Re^{i\sigma}) R i e^{i\sigma} d\sigma \Rightarrow \left|\int_{\Gamma_2} f(z)dz\right| \leq \int_0^\alpha |f(Re^{i\sigma})| R d\sigma \to 0$$

$$\Rightarrow \int_{\Gamma_2} f(z)dz \to 0$$

$$\int_{\Gamma_3} f(z)\,dz = \int_R^0 f(e^{i\alpha}t)\,e^{i\alpha}\,dt$$

$$= -e^{i\alpha}\int_0^R f(e^{i\alpha}t)\,dt \longrightarrow -e^{i\alpha}\int_0^\infty f(e^{i\alpha}t)\,dt$$

$$\Rightarrow \sum_i \int_{\Gamma_i} f(z)\,dz = 0 \Rightarrow \boxed{\int_0^\infty f(x)\,dx = e^{i\alpha}\int_0^\infty f(e^{i\alpha}x)\,dx}$$

b) Usando (1) con $f(z) = e^{-z^2}$ y $\alpha = \frac{\pi}{8}$

$$\Rightarrow \int_0^\infty e^{-x^2}\,dx = e^{i\frac{\pi}{8}} \cdot \int_0^\infty e^{-(e^{i\frac{\pi}{8}}t)^2}\,dt$$

$$= e^{i\frac{\pi}{8}} \int_0^\infty e^{-e^{i\frac{\pi}{4}} \cdot t^2}\,dt$$

$$= e^{i\frac{\pi}{8}} \int_0^\infty e^{-\frac{(1+i)}{\sqrt{2}} t^2}\,dt$$

$t = \sqrt[4]{2}\,u$

$dt = \sqrt[4]{2}\,du$

$$= e^{i\frac{\pi}{8}} \int_0^\infty e^{-(1+i)u^2} \sqrt[4]{2}\,du$$

$$= \sqrt[4]{2}\,e^{i\frac{\pi}{8}} \int_0^\infty e^{-u^2}\left(\cos(u^2) - i\sin(u^2)\right) du$$

$$\Rightarrow \frac{e^{-i\frac{\pi}{8}}}{\sqrt[4]{2}} \cdot \int_0^\infty e^{-x^2}\,dx = I_1 - iI_2$$

Usar $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi} \Rightarrow \int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

$\Rightarrow \frac{\sqrt{\pi}}{2} \cdot \frac{\cos(\pi/8)}{\sqrt[4]{2}} = I_1$ y $\frac{\sqrt{\pi}}{2} \frac{\sin(\pi/8)}{\sqrt[4]{2}} = I_2$

$\frac{\sqrt{\pi}}{4} \cdot \sqrt{\sqrt{2}+1} = I_1$ y $\frac{\sqrt{\pi}}{4} \sqrt{\sqrt{2}-1} = I_2$