P1 a) $\gamma(t) = t^2+it \implies \dot{\gamma}(t) = 2t+i$

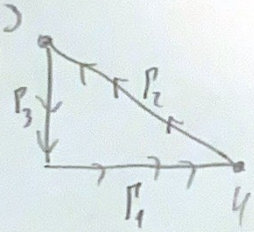
$$\int_0^2 \overline{(t^2+it)} \cdot (2t+i)\,dt = \int_0^2 (t^2-it)(2t+i)\,dt$$

$$= \int_0^2 (2t^3 - 2it^2 + it^2 + t)\,dt$$

$$= \int_0^2 2t^3 - it^2 + t\,dt$$

$$= \frac{t^4}{2} - \frac{i}{3}t^3 + \frac{t^2}{2}\Big|_0^2$$

$$= 8 - \frac{8}{3}i + 2 = \boxed{\left(10 - \tfrac{8}{3}i\right)}$$

b) 


$\gamma_1(t) = t \quad , t \in [0,4]$

$\dot{\gamma}_1(t) = 1$

$\gamma_2(t) = 4(1-t) + 3i(t) \quad , \quad \dot{\gamma}_2(t) = -4+3i \quad , t \in [0,1]$

$\gamma_3(t) = i(3-t) \quad , \dot{\gamma}_3(t) = -i \quad , \quad t \in [0,3]$

$$\int_{\Gamma_1} \bar{z}\,dz = \int_0^4 t\,dt = \frac{t^2}{2} = \boxed{8} \quad , \quad \int_{\Gamma_2} \bar{z}\,dz = \int_0^1 (4(1-t) - 3it)(-4+3i)\,dt$$

$$= \int_0^1 -16(1-t) - \cancel{12it} + 12i + \cancel{12it} + 9t\,dt$$

$$= \int_0^1 -16 + 12i + 16t + 9t\,dt$$

1

$$= -16t + 12it + 8t^2 + \frac{9}{2}t^2 \Big|_0^1$$

$$= -16 + 12i + 8 + \frac{9}{2} = \cancel{\boxed{-8 - \frac{15}{2}i}} \quad \boxed{-8 + \frac{9}{2} + 12i}$$

$$\int_{\Gamma_3} \bar{z}\,dz = \int_0^3 (-i)(3-t)(-i)\,dt$$

$$= \int_0^3 (t-3)\,dt$$

$$= \frac{(t-3)^2}{2}\Big|_0^3$$

$$= \boxed{-\frac{9}{2}}$$

$$\int_\Gamma \bar{z}\,dz = 8 + \left(-8 + \frac{9}{2}\right) + \left(-\frac{9}{2}\right) + 12i = \boxed{12i}$$

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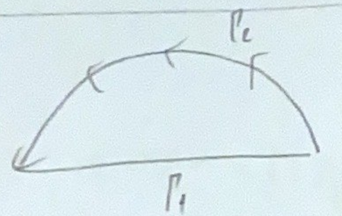
P2




$\Gamma_1: \gamma_1(t) = t, \ \dot{\gamma}_1(t) = 1, \ t \in [-1, 1]$

$\Gamma_2: \gamma_2(t) = e^{it}, \ \dot{\gamma}_2(t) = i e^{it}, \ t \in [0, \pi]$

$$\int_{\Gamma_1} z^2\,dz = \int_{-1}^1 t^2\,dt = \frac{t^3}{3}\Big|_{-1}^1 = \boxed{\frac{2}{3}}$$

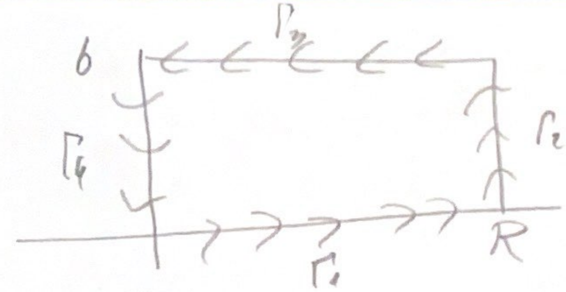
$$\int_{\Gamma_2} z^2\,dz = \int_0^\pi e^{2it} \cdot i e^{it}\,dt = i\int_0^\pi e^{3it}\,dt$$

$$= \frac{i}{3i} e^{3it}\Big|_0^\pi = \frac{1}{3} \cdot \left[e^{3\pi} - e^0\right]$$

$$= \boxed{-\frac{2}{3}}$$

$$\int_\Gamma z^2\,dz = \frac{2}{3} + \left(-\frac{2}{3}\right) = \boxed{0}$$

P3 $f(z)=e^{-z^2}$




Para esta pregunta se puede usar que $\displaystyle\int_{\square} e^{-z^2}dz = 0$

$\gamma_1(t)=t$ , $t\in[0,R]$
$\dot{\gamma}_1(t)=1$

$$\int_{\Gamma_1} e^{-z^2}dz = \int_0^R e^{-t^2}dt = I_1(R)$$

$\gamma_2(t)=R+it$ , $t\in[0,b]$
$\dot{\gamma}_2(t)=i$

$$\int_{\Gamma_2} e^{-z^2}dz = \int_0^b e^{-(R+it)^2}\, i\,dt = i\int_0^b e^{-R^2+t^2-2itR}dt$$

$$= ie^{-R^2}\int_0^b e^{t^2}\cdot e^{-2itR}dt = I_2(R)$$

$\gamma_3(t)=t+ib$ , $t\in[R,0]$
$\dot{\gamma}_3(t)=1$

$$\int_{\Gamma_3} e^{-z^2}dz = \int_R^0 e^{-(t+ib)^2}dt$$

$$= \int_R^0 e^{-t^2}\cdot e^{b^2}\cdot e^{-2ibt}dt$$

$$= e^{b^2}\int_R^0 e^{-t^2}(\cos(2bt) - i\sin(2bt))\,dt$$

$$= -e^{b^2}\int_0^R e^{-t^2}(\cos(2bt) - i\sin(2bt))\,dt$$

$\gamma_4(t)=it$ , $t\in[b,0]$
$\dot{\gamma}_4(t)=i$

$$\int_{\Gamma_4} e^{-z^2}dz = \int_b^0 e^{t^2}\, i\,dt = (-i)\int_0^b e^{t^2}dt$$

$$= I_4(R)$$

Tenemos $I_1(R) + I_2(R) + I_3(R) + I_4(R) = 0$

Tomando $R \to \infty$

$$I_1(R) = \int_0^\infty e^{-x^2} dx$$

$$|I_2(R)| \leq e^{-R^2} \int_0^b e^{t^2} \underbrace{|e^{-2iRt}|}_{\leq 1} dt \leq \underbrace{\underset{\searrow 0}{e^{-R^2}} \cdot \underbrace{\int_0^b e^{t^2} dt}_{\text{Finito}}}_{\searrow 0}$$

$$I_2(R) \to 0$$

$$I_3(R) \longrightarrow -\int_0^\infty e^{-x^2} \cos(2xb) dx + i\int_0^\infty e^{-x^2} \sin(2xb) dx$$

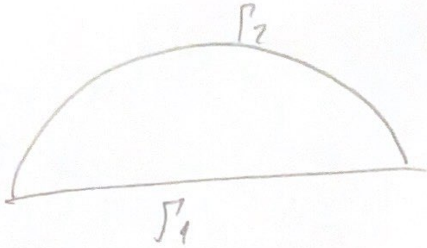
$$I_4(R) \longrightarrow -i \int_0^b e^{y^2} dy$$

Igualando Reales y complejos

$$\Rightarrow \int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-x^2} \cos(2xb) dx$$

$$\int_0^b e^{y^2} dy = \int_0^\infty e^{-x^2} \sin(2xb) dx \qquad \blacksquare$$

P4




$\gamma_1(t) = t \,,\ \gamma_1'(t) = 1 \,,\ t \in [-R, R]$

$\gamma_2(t) = Re^{it} \,,\ \gamma_2'(t) = iRe^{it} \,,\ t \in [0, \pi]$

$$\int_{\Gamma_1} \frac{\log(z+i)}{z^2+1}\,dz = \int_{-R}^{R} \frac{\log(t+i)}{t^2+1}\,dt = \int_{-R}^{R} \frac{\ln(\sqrt{t^2+1}) + i\,\mathrm{Arg}\left(\frac{1}{t}\right)}{t^2+1}\,dt$$

$$= \underbrace{\frac{1}{2}\int_{-R}^{R} \frac{\ln(t^2+1)}{t^2+1}\,dt}_{\text{Real}} + i \underbrace{\int_{-R}^{R} \frac{\mathrm{Arg}\left(\frac{1}{t}\right)}{t^2+1}\,dt}_{\text{Imaginari}}$$

$$\int_{\Gamma_2} \frac{\log(z^2+1)}{z^2+1}\,dz = \int_0^{\pi} \frac{\log\left(\sqrt{(R\cos(t))^2 + (R\sin(t)+1)^2}\right) + \mathrm{Arg}\left(R^2 e^{it}+1\right)}{R^2 e^{2it}+1}\,Re^{it}\,dt$$

Idea: Arriba orden $R$, Abajo orden $R^2$, va a 0

Acotando log ↓     Acotando ↓ denominador

$$|I_2| \leq \int_0^{\pi} \frac{\log(\sqrt{2R^2+1}) + 2\pi}{R^2-1}\,R\,dt$$

$$\leq \frac{\pi\left(\log(\sqrt{2R^2+1}) + 2\pi\right)}{R - \frac{1}{R}} \longrightarrow 0$$

$R$ gana $\ln(R^\alpha) = \alpha \ln(R)$

$$\Rightarrow I_1 + I_2 = \pi \ln(2) + i\frac{\pi^2}{2} \Rightarrow \frac{1}{2}\int_{-\infty}^{\infty} \frac{\ln(t^2+1)}{t^2+1}\,dt = \pi \ln(2)$$

Usando $I_1$ es par $\Rightarrow$ $\boxed{\int_0^{\infty} \frac{\ln(t^2+1)}{t^2+1}\,dt = \pi \ln(2)}$