P1 $h_1 = h_2 = h_3 = 1$, $\hat{u} = \hat{\imath}$, $\hat{v} = \hat{\jmath}$, $\hat{w} = \hat{k}$

1) $F_1 = y^2 \cos(x) + z^3$, $F_2 = 2y\sin(x) - 4$, $F_3 = 3xz^2 + 2z$

$$\text{rot}(F) = \left| \begin{array}{ccc:cc} \hat{\imath} & \hat{\jmath} & \hat{k} & \hat{\imath} & \hat{\jmath} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z & \partial/\partial x & \partial/\partial y \\ F_1 & F_2 & F_3 & F_1 & F_2 \end{array} \right|$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\imath} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

$$= 0\,\hat{\imath} + (\cancel{3z^2} - \cancel{3z^2})\,\hat{\jmath} + (\cancel{2y\cos(x)} - \cancel{2y\cos(x)})\,\hat{k}$$

$$= 0 \quad \checkmark$$

(irrotacional)

2) $\vec{G} = (G_1, G_2, G_3)$, $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$, $\hat{u} = \hat{\rho}$, $\hat{v} = \hat{\theta}$, $\hat{w} = \hat{k}$

$$G_1 = \frac{\rho z}{(\rho^2 + z^2)^{3/2}} , \quad G_2 = 0 , \quad G_3 = \frac{-\rho^2}{(\rho^2 + z^2)^{3/2}}$$

$$\text{rot}(G) = \frac{1}{\rho} \left| \begin{array}{ccc:cc} \hat{\rho} & \rho\hat{\theta} & \hat{k} & \hat{\rho} & \rho\hat{\theta} \\ \partial/\partial\rho & \partial/\partial\theta & \partial/\partial z & \partial/\partial\rho & \partial/\partial\theta \\ \frac{\rho z}{(\rho^2+z^2)^{3/2}} & 0 & \frac{-\rho^2}{(\rho^2+z^2)^{3/2}} & G_1 & 0 \end{array} \right|$$

$$= \frac{1}{\rho}\left(\Big[\underbrace{0}_{\frac{\partial G_3}{\partial \theta}=0} - \underbrace{0}_{G_2=0}\Big]\hat{\rho} + \left[\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial \rho}\right]\rho\hat{\theta} + \Big[\underbrace{0}_{G_2=0} - \underbrace{0}_{\frac{\partial G_1}{\partial \theta}=0}\Big]\hat{k} \right)$$

$$= \left[\frac{\rho(\rho^2+z^2)^{\frac{3}{2}} - 3(\rho^2+z^2)^{\frac{1}{2}}\rho z^2}{(\rho^2+z^2)^3} + \frac{2\rho(\rho^2+z^2)^{\frac{3}{2}} - 3(\rho^2+z^2)^{\frac{1}{2}}\rho^3}{(\rho^2+z^2)^3} \right]\hat{\theta}$$

$$= \frac{1}{(\rho^2+z^2)^3}\left((\rho^2+z^2)^{\frac{1}{2}}\rho\underbrace{\left[3\cancel{(\rho^2+z^2)}-\cancel{3\rho^2}-\cancel{3z^2}\right]}_{0}\right)\hat{\theta}$$

$$= 0 \quad \checkmark \quad \text{es irrotacional}$$

$\boxed{P2}$ $\operatorname{div}(\underbrace{f\vec{F}}) = \operatorname{div}(\vec{G})$

Con $\vec{G} = (fF_1, fF_2, fF_3)$

Vamos a hacerlo en cartesianas

$$\operatorname{div}(\vec{G}) = \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z}$$

$$= \frac{\partial f F_1}{\partial x} + \frac{\partial f F_2}{\partial y} + \frac{\partial f F_3}{\partial z}$$

$$= \frac{\partial f}{\partial x}\cdot F_1 + f\frac{\partial F_1}{\partial x} + \frac{\partial f}{\partial y}\cdot F_2 + f\frac{\partial F_2}{\partial y} + \frac{\partial f}{\partial z}\cdot F_3 + f\frac{\partial F_3}{\partial z}$$

$$= f\left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right] + \frac{\partial f}{\partial x}\cdot F_1 + \frac{\partial f}{\partial y}\cdot F_2 + \frac{\partial f}{\partial z}\cdot F_3$$

$$= f\cdot \operatorname{div}(\vec{F}) + \nabla f \cdot \vec{F}$$

■

Para concluir basta tomar $\vec{F} = \nabla f$ y usar que $\Delta f = \operatorname{div}(\nabla f)$

$$\Rightarrow \operatorname{div}(f\nabla f) = f\cdot\Delta f + \nabla f\cdot\nabla f$$

$$= f\Delta f + \|\nabla f\|^2$$

P3

$x(\varepsilon,\eta,\phi) = \varepsilon\eta\cos(\phi)$

$y(\varepsilon,\eta,\phi) = \varepsilon\eta\sin(\phi)$

$z(\varepsilon,\eta,\phi) = \frac{1}{2}(\eta^2-\varepsilon^2)$

$\vec{r} = (x, y, z)$

$$\vec{\varepsilon} = \frac{d\vec{r}}{d\varepsilon} = (\eta\cos\phi, \eta\sin\phi, -\varepsilon) \Rightarrow \|\vec{\varepsilon}\| = \boxed{h_\varepsilon = \sqrt{\eta^2+\varepsilon^2}}$$

$$\boxed{\hat{\varepsilon} = \frac{1}{\sqrt{\eta^2+\varepsilon^2}}(\eta\cos\phi, \eta\sin\phi, -\varepsilon)}$$

$$\vec{\eta} = (\varepsilon\cos\phi, \varepsilon\sin\phi, \eta) \Rightarrow \|\vec{\eta}\| = \boxed{h_\eta = \sqrt{\eta^2+\varepsilon^2}}$$

$$\boxed{\hat{\eta} = \frac{1}{\sqrt{\eta^2+\varepsilon^2}}(\varepsilon\cos\phi, \varepsilon\sin\phi, \eta)}$$

$$\vec{\phi} = (-\varepsilon\eta\sin\phi, \varepsilon\eta\cos\phi, 0) \Rightarrow \|\vec{\phi}\| = \boxed{h_\phi = \varepsilon\eta}$$

$$\boxed{\hat{\phi} = (-\sin\phi, \cos\phi, 0)}$$

$$\boxed{\nabla f = \frac{1}{\sqrt{\eta^2+\varepsilon^2}}\left(\frac{\partial f}{\partial \varepsilon}\hat{\varepsilon} + \frac{\partial f}{\partial \eta}\hat{\eta}\right) + \frac{1}{\varepsilon\eta}\frac{\partial f}{\partial \phi}\hat{\phi}}$$

(Solo reemplazar)
Puede quedar $\hat{\varepsilon}, \hat{\eta}, \hat{\phi}$ expresado si se calculan

$$\Delta f = \frac{1}{(\eta^2+\varepsilon^2)\varepsilon\eta}\left[\eta\frac{\partial f}{\partial \varepsilon} + \varepsilon\frac{\partial f}{\partial \eta} + \varepsilon\eta\left(\frac{\partial^2 f}{\partial \varepsilon^2} + \frac{\partial^2 f}{\partial \eta^2}\right) + \left(\frac{\varepsilon^2+\eta^2}{\varepsilon\eta}\right)\frac{\partial^2 f}{\partial \phi^2}\right]$$

(Hay que hacer las derivadas para llegar a esto)

PD: A veces no dan Δf y hay que usar $\Delta f = \mathrm{div}(\nabla f)$

P3 Extra

Para probar que son ortogonales, basta que

$$\begin{cases} \hat{\xi}\cdot\hat{\eta}=0 \\ \hat{\xi}\cdot\hat{\phi}=0 \\ \hat{\eta}\cdot\hat{\phi}=0 \end{cases}$$

o

$$\hat{\xi}\times\hat{\eta}=\pm\hat{\phi}$$

Opción 1

$$\hat{\xi}\cdot\hat{\eta}=\frac{1}{(\eta^2+\xi^2)}\Big(\underbrace{\eta\xi\cos^2\phi+\eta\xi\sin^2\phi}_{\eta\xi}-\eta\xi\Big)=0$$

$$\hat{\xi}\cdot\hat{\phi}=\frac{1}{(\eta^2+\xi^2)^{1/2}}\Big(-\cancel{\eta\cos\phi\sin\phi}+\cancel{\eta\cos\phi\sin\phi}+0\Big)=0$$

$$\hat{\eta}\cdot\hat{\phi}=\frac{1}{(\eta^2+\xi^2)^{1/2}}\Big(-\cancel{\xi\cos\phi\sin\phi}+\cancel{\xi\cos\phi\sin\phi}+0\Big)=0$$

Opción 2

$$\hat{\xi}\times\hat{\eta}=\frac{1}{(\eta^2+\xi^2)}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k} \\ \eta\cos\phi & \eta\sin\phi & -\xi \\ \xi\cos\phi & \xi\sin\phi & \eta\end{vmatrix}$$

$$=\frac{1}{(\eta^2+\xi^2)}\Big[(\eta^2\sin\phi-\xi^2\sin\phi)\hat{i}+(-\xi^2\cos\phi-\eta^2\cos\phi)\hat{j}+\overbrace{(\cancel{\eta\xi\cos\phi\sin\phi}-\cancel{\eta\xi\cos\phi\sin\phi})}^{0}\hat{k}\Big]$$

$$=(\sin\phi,-\cos\phi,0)=-\hat{\phi}$$

Cualquiera de las 2 sirve

P4 | Vamos a ver el rotor en cilíndricas $\hat{\rho}, \hat{\theta}, \hat{k}$, $h_\rho = h_z = 1$
$h_\theta = \rho$

a) $rot(\vec{F}) = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\theta} & \hat{k} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_\rho(\rho) & 0 & 0 \end{vmatrix}\begin{matrix} \hat{\rho} & \rho\hat{\theta} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} \\ F_\rho(\rho) & 0 \end{matrix}$

$$= \cancelto{0}{\left(\frac{\partial 0}{\partial \theta} - \frac{\partial 0}{\partial z}\right)}\hat{\rho} + \left(\cancelto{0}{\frac{\partial F_\rho(\rho)}{\partial z}} - \cancelto{0}{\frac{\partial 0}{\partial \rho}}\right)\rho\hat{\theta} + \left(\frac{\partial 0}{\partial \rho} - \cancelto{0}{\frac{\partial F_\rho(\rho)}{\partial \theta}}\right)\hat{k}$$

$= 0$ (Son derivadas de 0 o derivar algo que no depende de eso)

b) $div(\vec{F}) = \frac{1}{\rho}\left(\frac{\partial(F_\rho \cdot \rho \cdot 1)}{\partial \rho} + \cancelto{0}{\frac{\partial(0 \cdot 1 \cdot 1)}{\partial \theta}} + \cancelto{0}{\frac{\partial(0 \cdot 1 \cdot \rho)}{\partial z}}\right)$

$= \frac{1}{\rho}\left(\frac{\partial(F_\rho(\rho) \cdot \rho)}{\partial \rho}\right)$

c) $div(\vec{F}) = 0 = \frac{1}{\rho}\left(\frac{\partial(F_\rho(\rho) \cdot \rho)}{\partial \rho}\right)$

$\Leftrightarrow F_\rho(\rho) \cdot \rho = K$ (No puede depender de θ, z ¿Por qué? Simetría)

$\Leftrightarrow F_\rho(\rho) = \frac{K}{\rho}$

$\Leftrightarrow \vec{F}(\rho, \theta, z) = \frac{K}{\rho}\hat{\rho}$ / Usando que $\vec{F} = F_\rho$

Propuestos

$\boxed{P1}$ Notar que $\nabla \times \vec{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{\imath} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k}$

$$\vec{v} \times (\nabla \times \vec{v}) = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} & \hat{\imath} & \hat{\jmath} \\ v_1 & v_2 & v_3 & v_1 & v_2 \\ \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) & \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) & \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) & \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) & \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) \end{bmatrix}$$

$$\left.\begin{aligned} &= v_2\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) - v_3\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\hat{\imath} \\ &v_3\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - v_1\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{\jmath} \\ &v_1\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) - v_2\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{k} \end{aligned}\right\} (1)$$

$$\left.\begin{aligned} \frac{\nabla}{2}(v_1^2 + v_2^2 + v_3^2) = &\left(\frac{\partial v_1}{\partial x} \cdot v_1 + \frac{\partial v_2}{\partial x} v_2 + \frac{\partial v_3}{\partial x} \cdot v_3\right)\hat{\imath} \\ &+ \left(\frac{\partial v_1}{\partial y} v_1 + \frac{\partial v_2}{\partial y} \cdot v_2 + \frac{\partial v_3}{\partial y} v_3\right)\hat{\jmath} \\ &+ \left(\frac{\partial v_1}{\partial z} v_1 + \frac{\partial v_2}{\partial z} \cdot v_2 + \frac{\partial v_3}{\partial z} \cdot v_3\right)\hat{k} \end{aligned}\right\} (2)$$

$$(2)-(1) = \left(v_1 \frac{\partial v_1}{\partial x} + v_2 \frac{\partial v_1}{\partial y} + v_3 \frac{\partial v_1}{\partial z}\right)\hat{\imath} + \left(v_1 \frac{\partial v_2}{\partial x} + v_2 \frac{\partial v_2}{\partial y} + v_3 \frac{\partial v_2}{\partial z}\right)\hat{\jmath}$$
$$+ \left(v_1 \frac{\partial v_3}{\partial x} + v_2 \frac{\partial v_3}{\partial y} + v_3 \frac{\partial v_3}{\partial z}\right)\hat{k} = \nabla \vec{v} \cdot \vec{v}$$

Considerando

$$\nabla\vec{v} = \begin{bmatrix} \frac{\partial v_1}{\partial x} & \frac{\partial v_1}{\partial y} & \frac{\partial v_1}{\partial z} \\ \frac{\partial v_2}{\partial x} & \frac{\partial v_2}{\partial y} & \frac{\partial v_2}{\partial z} \\ \frac{\partial v_3}{\partial x} & \frac{\partial v_3}{\partial y} & \frac{\partial v_3}{\partial z} \end{bmatrix}$$

$$\nabla\vec{v}\cdot v = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{bmatrix} v_1 \frac{\partial v_1}{\partial x} + v_2 \frac{\partial v_1}{\partial y} + v_3 \frac{\partial v_1}{\partial z} \\ v_1 \frac{\partial v_2}{\partial x} + v_2 \frac{\partial v_2}{\partial y} + v_3 \frac{\partial v_2}{\partial z} \\ v_1 \frac{\partial v_3}{\partial x} + v_2 \frac{\partial v_3}{\partial z} + v_3 \frac{\partial v_3}{\partial z} \end{bmatrix}$$

Que justamente es $\frac{1}{2}\nabla(v_1^2+v_2^2+v_3^2) - \vec{v}\times(\nabla\times\vec{v})$

b) Irrotacional $\Rightarrow$ $\nabla\times\vec{v}=0$ (1)

Incompresible $\Rightarrow$ ρ cte $\Rightarrow$ $\nabla\rho=0$ (2)

(1) $\Rightarrow$ $\nabla\vec{v}\cdot\vec{v} = \frac{1}{2}\nabla(v_1^2+v_2^2+v_3^2)$

Bernoulli $\Rightarrow$ $\rho(\nabla\vec{v}\cdot\vec{v}) + \nabla p \overset{(1)}{=} \frac{\rho}{2}\left(\nabla(v_1^2+v_2^2+v_3^2)\right) + \nabla p = -\rho g\hat{k}$

ρ puede entrar $\Rightarrow$

$$\nabla\left(\frac{\rho}{2}(v_1^2+v_2^2+v_3^2) + p\right) = -\rho g\hat{k}$$

Hay 3 ecuaciones como sólo sobrevive la derivada respecto a z

$\overset{\int dz}{\Rightarrow}$ $\frac{\rho}{2}(v_1^2+v_2^2+v_3^2) + p + \rho g z = C(x,y) \overset{\text{sol. e}}{\Rightarrow} C(x,y) = C$, pues $\frac{\partial C}{\partial x} = \frac{\partial C}{\partial y} = 0$

$= C$

$$\boxed{2} \quad \vec{\sigma} = \frac{\partial r}{\partial \sigma} = \left(\frac{a \sinh(\tau) \sin(\sigma)}{(\cosh(\tau) - \cos(\sigma))^2} , \frac{a(\cos(\sigma)\cosh(\tau) - 1)}{(\cosh(\tau) - \cos(\sigma))^2} \right)$$

$$h_\sigma = \|\vec{\sigma}\| = \frac{a}{(\cosh(\tau) - \cos(\sigma))^2} \left[\sinh(\tau)^2 \sin(\sigma)^2 + \cos(\sigma)^2 \cosh(\tau)^2 - 2\cosh(\tau)\cos(\sigma) + 1 \right]^{1/2}$$

$$= \frac{a}{(\cosh(\tau) - \cos(\sigma))^2} \left[\cosh(\tau)^2 \sin(\sigma)^2 - \sin(\sigma)^2 + \cosh(\tau)^2 \cos(\sigma)^2 - 2\cosh(\tau)\cos(\sigma) + 1 \right]^{1/2}$$

$$= \frac{a}{(\cosh(\tau) - \cos(\sigma))^2} \underbrace{\left[\cosh(\tau)^2 - 2\cosh(\tau)\cos(\sigma) + \cos(\sigma)^2 \right]^{1/2}}_{|\cosh(\tau) - \cos(\sigma)|}$$

$$= \frac{a}{|\cosh(\tau) - \cos(\sigma)|}$$

Notar τ^0 $\cosh(\tau) \geq 1 \geq \cos(\sigma)$

$$\boxed{= \frac{a}{\cosh(\tau) - \cos(\sigma)}}$$

$$\Rightarrow \boxed{\hat{\sigma} = \left(\frac{\sinh(\tau)\sin(\sigma)}{\cosh(\tau) - \cos(\sigma)} , \frac{\cos(\sigma)\cosh(\tau) - 1}{\cosh(\tau) - \cos(\sigma)} \right)}$$

$$\vec{\tau} = \frac{\partial r}{\partial \tau} = \left(\frac{a(1 - \cosh(\tau)\cos(\sigma))}{(\cosh(\tau) - \cos(\sigma))^2} , \frac{a \sin(\sigma)\sinh(\tau)}{(\cosh(\tau) - \cos(\sigma))^2} \right)$$

$h_\tau = \|\vec{\tau}\| =$ es lo mismo pues conmuta $\sqrt{a^2 + b^2} = \sqrt{b^2 + a^2}$

$$\boxed{= \frac{a}{\cosh(\tau) - \cos(\sigma)}}$$

$$\Rightarrow \boxed{\hat{\tau} = \left(\frac{1 - \cosh(\tau)\cos(\sigma)}{\cosh(\tau) - \cos(\sigma)} , \frac{\sin(\sigma)\sin(\tau)}{\cosh(\tau) - \cos(\sigma)} \right)}$$

Para usar formula ∇f considerar $h_w = 1$ y $\frac{\partial}{\partial w} = 0$

$$\nabla f = \left(\frac{\cosh(\tau) - \cos(\sigma)}{a} \right) \left(\frac{\partial f}{\partial \sigma} \hat{\sigma} + \frac{\partial f}{\partial \tau} \hat{\tau} \right)$$

$$\Delta f = \frac{(\cosh(\tau) - \cos(\sigma))^2}{a^2} \left(\frac{\partial^2 f}{\partial \sigma^2} + \frac{\partial^2 f}{\partial \tau^2} \right)$$

Obs: Notar que Δf se puede calcular solo asumiendo la indicación.