P7. (a) Sea $f : \mathbb{R}^2 \to \mathbb{R}$ tal que $f \in C^2$, que satisface la ecuación de Laplace:

$$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = 0$$

Sean $u(x, y), v(x, y)$ funciones diferenciables que satisfacen las ecuaciones de Cauchy-Riemann, es decir:

$$\frac{\partial u}{\partial x}(x, y) = \frac{\partial v}{\partial y}(x, y) \ , \ \frac{\partial u}{\partial y}(x, y) = -\frac{\partial v}{\partial x}(x, y)$$

Demuestre que la función:

$$g(x, y) = f(u(x, y), v(x, y))$$

también satisface la ecuación de Laplace.

Pdq : $\dfrac{\partial^2 g}{\partial x^2} + \dfrac{\partial^2 g}{\partial y^2} = 0$

→ Calculamos las derivadas parciales

$$\cdot \frac{\partial g}{\partial x} = \frac{\partial f}{\partial u}(u, v) \cdot \frac{\partial u}{\partial x}(x, y)$$

$$+ \frac{\partial f}{\partial v}(u, v) \cdot \frac{\partial u}{\partial x}(x, y)$$

$$\rightarrow \frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}(u, v) \cdot \frac{\partial u}{\partial x}(x, y)\right.$$

$$\left. + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial v}(u, v) \cdot \frac{\partial v}{\partial x}(x, y)\right)\right)$$

$$\rightarrow \frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}(u(x, y), v(x, y))\right)\frac{\partial u}{\partial x}(x, y)$$

$$+ \frac{\partial f}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial v}(u(x, y), v(x, y))\right)\frac{\partial v}{\partial x}$$

$$+ \frac{\partial f}{\partial v}(u, v)\frac{\partial^2 v}{\partial x^2}$$

$$\rightarrow \frac{\partial^2 g}{\partial x^2} = \left[\frac{\partial^2 f}{\partial u^2}\cdot\frac{\partial u}{\partial x} + \frac{\partial^2 f}{\partial v \partial u}\frac{\partial v}{\partial x}\right]\frac{\partial u}{\partial x}$$

$$+ \frac{\partial f}{\partial u}\frac{\partial^2 u}{\partial x^2} + \left[\frac{\partial^2 y}{\partial u \partial v}\cdot\frac{\partial u}{\partial x} + \frac{\partial^2 y}{\partial v^2}\frac{\partial v}{\partial x}\right]\frac{\partial v}{\partial x}$$

$$+ \frac{\partial f}{\partial v}\cdot\frac{\partial^2 v}{\partial x^2} \qquad (2)$$

por otro lado, con respecto a y.

$$\frac{\partial g}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$\rightarrow \frac{\partial^2 g}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial u}(u(x,y), v(x,y))\right)\frac{\partial u}{\partial y}$$

$$+ \frac{\partial f}{\partial u}\cdot\frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial v}(u(x,y), v(x,y))\right)\frac{\partial v}{\partial y}$$

$$+ \frac{\partial f}{\partial v}\frac{\partial^2 v}{\partial y^2}$$

$$\rightarrow \frac{\partial^2 g}{\partial y^2} = \left[\frac{\partial^2 f}{\partial u^2}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial v \partial u}\frac{\partial v}{\partial y}\right]\frac{\partial u}{\partial y} + \frac{\partial f}{\partial u}\frac{\partial^2 u}{\partial y^2} + \left[\frac{\partial^2 f}{\partial u \partial v}\frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial y}\right]\frac{\partial v}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial^2 v}{\partial y^2} \qquad (1)$$

Por cauchy - Riemann

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

$\hookrightarrow$ esto se obtiene derivando denuevo la igualdad del enunciado

Sumando ambas igualdades

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

(por schartz)

$\rightarrow$ Analogamente:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Entonces:

$$\frac{\partial f}{\partial u}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)=0$$

$\longrightarrow$ Remplazando:

$$\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}=\left[\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial x}+\frac{\partial^2 y}{\partial v\partial u}\frac{\partial v}{\partial x}\right]\frac{\partial u}{\partial x}$$

$$+\left[\frac{\partial y}{\partial u\partial v}\cdot\frac{\partial u}{\partial x}+\frac{\partial^2 y}{\partial v\partial u}\cdot\frac{\partial v}{\partial y}\right]\frac{\partial u}{\partial y}$$

$$+\left[\frac{\partial^2 f}{\partial u\partial v}\frac{\partial v}{\partial y}+\frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial y}\right]\frac{\partial v}{\partial y}$$

Como f es de clase C^2, se tiene que:

$$\frac{\partial^2 f}{\partial v\partial u}\cdot\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial x}+\frac{\partial^2 f}{\partial u\partial v}\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}$$

$$+\frac{\partial^2 f}{\partial v\partial u}\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}+\frac{\partial^2 f}{\partial u\partial v}\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}$$

$$=2\cdot\frac{\partial^2 f}{\partial v\partial u}\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}\right)$$

Por las condiciones de C - R
(cauchy - Riemann) (multiplicando)

$$\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}$$

Trabajando la expresion:

$$\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y} = 0$$

$$2\cdot\frac{\partial^2 f}{\partial v\partial u}\left(\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}\right) = 0$$

Además notemos que:

Por las condicion de C - R

$$\left(\frac{\partial u}{\partial x}\right)^2 = \left(\frac{\partial v}{\partial y}\right)^2, \quad \left(\frac{\partial v}{\partial x}\right)^2 = \left(\frac{\partial u}{\partial y}\right)^2$$

luego, tendremos

$$\left(\frac{\partial u}{\partial x}\right)^2\underbrace{\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right)} + \left(\frac{\partial u}{\partial y}\right)^2\underbrace{\left(\frac{\partial^2 f}{\partial v^2} + \frac{\partial^2 f}{\partial u^2}\right)} = 0$$

Pues $\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = 0$

entonces :

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0$$

(b) Sea $f : \mathbb{R}^2 \to \mathbb{R}$ una función de clase C^2 que satisface:

$$\frac{\partial^2 f}{\partial x^2}(x,y) - \frac{\partial^2 f}{\partial y^2} = 0, \forall (x,y) \in \mathbb{R}^2$$

Muestre que una función $g : \mathbb{R}^2 \to \mathbb{R}$ tal que:

$$f(x,y) = g(u(x,y), v(x,y)) = g(x+y, x-y)$$

Satisface la ecuación:

$$\frac{\partial^2 g}{\partial u \partial v}(u,v) = 0$$

notamos que

$$U(x,y) = x+y \quad , \quad v(x,y) = x-y$$

entonces:

- $$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial U} \cdot \frac{\partial U}{\partial x} + \frac{\partial g}{\partial v} \frac{\partial v}{\partial x}$$

$$= \frac{\partial g}{\partial x} + \frac{\partial g}{\partial v}$$

→ derivando nuevamente:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 g}{\partial U^2} \cdot \frac{\partial U}{\partial x} + \frac{\partial^2 g}{\partial v \partial U} \cdot \frac{\partial v}{\partial x}$$

$$+ \frac{\partial^2 g}{\partial U \partial v} \cdot \frac{\partial U}{\partial x} + \frac{\partial^2 g}{\partial v^2} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 g}{\partial U^2} + \frac{\partial^2 g}{\partial v \partial U} + \frac{\partial^2 g}{\partial U \partial v} + \frac{\partial^2 g}{\partial v^2}$$

Analogamente:

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial u} - \frac{\partial g}{\partial v}$$

$$\rightarrow \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 g}{\partial u^2} - \frac{\partial^2 g}{\partial v \partial u} - \frac{\partial^2 g}{\partial u \partial v} + \frac{\partial^2 g}{\partial v^2}$$

luego, como

$$\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} = 0$$

Remplazando lo obtenido, llegamos a:

$$2\left(\frac{\partial^2 g}{\partial v \partial u} + \frac{\partial^2 g}{\partial u \partial v}\right) = 0$$

Como f es C^2, entonces u y v son C^2 y por lo tánto g es $C2$

luego, por swartz:

$$\frac{\partial^2 g}{\partial v \partial u} = \frac{\partial^2 g}{\partial u \partial v}$$

por lo tanto

$$y \quad \frac{\partial^2 g}{\partial u \partial v} = 0$$

$$\rightarrow \quad \frac{\partial^2 g}{\partial u \partial v} = 0 \qquad \blacksquare$$