P1

$$(*) = \begin{cases} u_{tt} = u_{xx} + 1, \quad x \in (0, \pi/2), \ t > 0 & (EQ) \\ u(0,t) = 1, \quad u_x(\pi/2, t) = -\pi/2 & (CB) \\ u(x,0) = -x^2 + \frac{1}{2}\pi x, \quad u_t(x,0) = 0, \ x \in (0, \pi/2) & (CI) \end{cases}$$

a) Consideramos el cambio de variables $\omega(x,t) = u(x,t) + f(x)$ $\Rightarrow u(x,t) = \omega(x,t) - f(x)$. Como sabemos que u es solución de $(*)$, reemplazando en (EQ):

$$u_{tt} = u_{xx} + 1 \overset{u=w-f}{\Longrightarrow} w_{tt} - \cancel{f_{tt}} = w_{xx} - \underbrace{f_{xx}}_{f''(x)} + 1$$

0 f no depende de t

Queremos llegar a una (EQ') igual a $w_{tt} = w_{xx}$, por lo tanto:

$$w_{tt} = w_{xx} \underbrace{- f''(x) + 1}_{\text{queremos que sea 0 para llegar a lo que queremos}}$$

desde integ. $\Rightarrow f''(x) = 1$

$\int dx + c_1$

$f'(x) = x + c_1$

$\int dx + c_2$

$$f(x) = \frac{x^2}{2} + c_1 x + c_2$$

Reemplazando $u = w - f$ en (CB):

$$u(0,t) = 1 \Rightarrow w(0,t) - f(0) = 1$$

$$w(0,t) = \underbrace{1 + f(0)}_{=0 \text{ por lo que se quiere llegar}}$$

$$\Rightarrow f(0) = -1$$

$$u_x(\pi/2,t)=-\pi/2 \Rightarrow w_x(\pi/2,t)-f'(\pi/2)=-\pi/2$$
$$\underbrace{w_x(\pi/2,t)}_{}=\underbrace{f'(\pi/2)-\pi/2}_{=0}$$

$$\Rightarrow f'(\pi/2)=\pi/2$$

Evaluando f en los ptos que encontramos:

$$f(x)=\frac{x^2}{2}+c_1x+c_2 \wedge f'(x)=x+c_1$$
$$f(0)=-1 \Rightarrow \boxed{c_2=-1}$$
$$f'(\pi/2)=\pi/2 \Rightarrow \cancel{\pi/2}+c_1=\cancel{\pi/2}$$
$$\Rightarrow \boxed{c_1=0}$$

Por último falta comprobar que $f(x)=\frac{x^2}{2}-1$ comprueba las (CI') para $w(x,t)$:

$$u(x,0)=-x^2+\frac{1}{2}\pi x \Rightarrow w(x,0)-f(x)=-x^2+\frac{1}{2}\pi x$$
$$\Rightarrow w(x,0)=-x^2+\frac{\pi}{2}x+\frac{x^2}{2}-1$$
$$=-\frac{x^2}{2}+\frac{\pi}{2}x-1 \quad ✓$$

$$u_t(x,0)=0 \Rightarrow w_t(x,0)-\underbrace{\cancel{f_t(x)}}_{0}=0 \quad ✓$$

$\therefore$ El cv $u(x,t)=w(x,t)+f(x)$ con $f(x)=\frac{x^2}{2}-1$ transforma (*) en (**).

$$(**)=\begin{cases} w_{tt}=w_{xx},\ x\in(0,\pi/2),\ t>0 & (EQ') \\ w(0,t)=0,\ w_x(\pi/2,t)=0 & (CB') \\ w(x,0)=-\frac{x^2}{2}+\frac{\pi}{2}x-1,\ w_t(x,0)=0,\ x\in[0,\frac{\pi}{2}] & (CI') \end{cases}$$

b) Si $w(x,t)=M(x)N(t)$, reemplazando en (EQ'):

$$w_{tt}=w_{xx} \Rightarrow N''(t)M(x)=M''(x)N(t)$$

$$\Rightarrow \underbrace{\frac{N''(t)}{N(t)}}_{\text{Sólo depende de } t} = \underbrace{\frac{M''(x)}{M(x)}}_{\text{'' de } x} \equiv -\lambda$$

para que calce con lo que hay que llegar por enunciado

Por lo tanto

$$\frac{M''(x)}{M(x)} = -\lambda$$

$$\Rightarrow \boxed{M''(x) + \lambda M(x) = 0}$$

Ahora reemplazando en (CB'):

$$w(0,t) = 0 \Rightarrow M(0)\cancel{N(t)} = 0 \quad / \quad N(t) \neq 0$$

de lo contrario sol trivial

$$\Rightarrow \boxed{M(0) = 0}$$

$$w_x(\tfrac{\pi}{2}, t) = 0 \Rightarrow M'(\tfrac{\pi}{2})\cancel{N(t)} = 0 \quad / \quad N(t) \neq 0$$

$$\Rightarrow \boxed{M'(\tfrac{\pi}{2}) = 0}$$

Resolviendo la EDO:

$$M''(x) + \lambda M(x) = 0$$

Ansatz $M(x) = e^{sx}$, reemplazando:

$$s^2 \cancel{e^{sx}} + \lambda \cancel{e^{sx}} = 0$$

$$s^2 + \lambda = 0$$

$$\Rightarrow \boxed{s = \pm\sqrt{-\lambda}}$$

$$\Rightarrow M(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$$

Comprobemos $M(0) = 0$ y $M'(\frac{\pi}{2}) = 0$

$$M(0) = 0 \Rightarrow A\cancel{e^0} + B\cancel{e^0} = 0$$

$$\Rightarrow \boxed{A = -B}$$

$$M'\left(\frac{\pi}{2}\right)=0 \Rightarrow A\sqrt{-\lambda}\,e^{\sqrt{-\lambda}\frac{\pi}{2}} \underbrace{+A}_{=B}(+\sqrt{-\lambda})e^{-\sqrt{-\lambda}\frac{\pi}{2}}=0$$

$$\Rightarrow A\left(e^{\sqrt{-\lambda}\frac{\pi}{2}}+e^{-\sqrt{-\lambda}\frac{\pi}{2}}\right)=0$$

$$\Rightarrow \underbrace{\cancel{A=0}}_{\text{lleva a la sol trivial}} \;\vee\; \underbrace{e^{\sqrt{-\lambda}\frac{\pi}{2}}+e^{-\sqrt{-\lambda}\frac{\pi}{2}}=0}_{\text{soluciones sólo para } \lambda>0}$$

(caso $\lambda<0$ no puede satisfacer $e^{\sqrt{-\lambda}\frac{\pi}{2}}+e^{-\sqrt{-\lambda}\frac{\pi}{2}}=0$)

$\lambda>0$ | Como tenemos una raíz negativa la sol pasa a ser compuesta por exponenciales complejas

$$M(x)=Ae^{i\sqrt{\lambda}x}+Be^{-i\sqrt{\lambda}x}$$

Además antes mostramos que $A=-B$ y $e^{i\sqrt{\lambda}\frac{\pi}{2}}+e^{-i\sqrt{\lambda}\frac{\pi}{2}}=0$. Por lo tanto:

$$\underset{A=-B}{\Rightarrow} M(x)=A\left(e^{i\sqrt{\lambda}x}-e^{-i\sqrt{\lambda}x}\right)$$

$$=\tilde{A}\sin(\sqrt{\lambda}x)$$

$$\frac{e^{i\theta}-e^{-i\theta}}{2i}=\sin\theta$$

$$\tilde{A}=\frac{A}{2i}$$

$$\Rightarrow e^{i\sqrt{\lambda}\frac{\pi}{2}}+e^{-i\sqrt{\lambda}\frac{\pi}{2}}=0$$

$$\frac{1}{2}\Bigg(\; \underbrace{\frac{e^{i\sqrt{\lambda}\frac{\pi}{2}}+e^{-i\sqrt{\lambda}\frac{\pi}{2}}}{2}}_{=\cos\left(\sqrt{\lambda}\frac{\pi}{2}\right)}=0$$

$$\cos\left(\sqrt{\lambda}\,\frac{\pi}{2}\right)=0$$

$$\Rightarrow \sqrt{\lambda}\,\frac{\cancel{\pi}}{2}=n\cancel{\pi}+\frac{\cancel{\pi}}{2}\quad, n\geq 0$$

$$\boxed{\lambda_n=(2n+1)^2}\quad, n\geq 0$$

$$\Rightarrow M(x)=\tilde{A}\sin\big((2n+1)x\big) \qquad M(x)=\tilde{A}\sin(\sqrt{\lambda}x)$$

c) Resolviendo la otra EDO ahora que sabemos que $\lambda_n = (2n+1)^2 > 0$:

$$\frac{N''(t)}{N(t)} = -\lambda$$

$$N''(t) + \lambda N(t) = 0$$

$$\underline{N''(t) + (2n+1)^2 N(t) = 0}$$

oscilador armonico

$$\Rightarrow N_n(t) = A_n \cos((2n+1)t) + B_n \sin((2n+1)t)$$

Por ppio de superposición

$$w(x,t) = \sum_{n=0}^{\infty} M_n(x) N_n(t)$$

$$\Rightarrow \underline{w(x,t) = \sum_{n=1}^{\infty} [A_n \cos(2n+1)t + B_n \sin((2n+1)t)] \sin((2n+1)x)}$$

d) Aplicando CI $w_t(x,0) = 0$:

$$w_t(x,0) = 0 \Rightarrow N'(0)\, \cancel{M(x)} = 0$$

$$\Rightarrow \underbrace{\cancel{-A_n(2n+1)\sin(0)}}_{0} + B_n(2n+1)\overset{1}{\cancel{\cos(0)}} = 0$$

$$\Rightarrow B_n \overset{\neq 0}{\cancel{(2n+1)}} = 0$$

$$\Rightarrow B_n = 0$$

$$\Rightarrow N(t) = A_n \cos((2n+1)t)$$

Y aplicando la otra CI:

$$w(x,0) = -\frac{x^2}{2} + \frac{\pi}{2}x - 1$$

$$\Rightarrow \sum_{n=0}^{\infty} A_n \underbrace{\cos(\cancel{(2n+1)} \cdot 0)}_{1} \sin((2n+1)x) = -\frac{x^2}{2} + \frac{\pi}{2}x - 1$$

$$\underbrace{\sum_{n=0}^{\infty} A_n \sin((2n+1)x) = -\frac{x^2}{2} + \frac{\pi}{2}x - 1}_{\text{serie de senos de } g(x) = -\frac{x^2}{2} + \frac{\pi}{2} - 1}$$

Multiplicando e integrando por $\left[\int_0^{\pi/2} \sin(mx)\,dx\right]$ a ambos lados de la igualdad

$2n+1=n'$

$$\int_0^{\pi/2} \sin(mx)\sum_{n=0}^{\infty} A_n \sin((2n+1)x)\,dx = \int_0^{\pi/2} g(x)\sin(mx)\,dx$$

$$\sum_{n=0}^{\infty} A_n \underbrace{\int_0^{\pi/2} \sin(mx)\sin(n'x)\,dx}_{=\begin{cases}\frac{\pi/2}{2} & m=n'\\ 0 & m\neq 0\end{cases}} = \int_0^{\pi/2} g(x)\sin(mx)\,dx$$

Como serie tiene un único término no nulo $m=n'$, entonces

$$A_{n'}\,\frac{\pi/2}{2} = \int_0^{\pi/2} g(x)\sin(n'x)\,dx$$

$$A_{n'} = \frac{4}{\pi}\int_0^{\pi/2} g(x)\sin(n'x)\,dx$$

$$= \frac{4}{\pi}\left[\int_0^{\pi/2} -\frac{x^2}{2}\sin(n'x)\,dx + \int_0^{\pi/2}\frac{\pi}{2}x\sin(n'x)\,dx - \int_0^{\pi/2}\sin(n'x)\,dx\right]$$

$n'x=u$
$dx=\frac{du}{n'}$
$x=\frac{u}{n'}$
$x=\frac{\pi}{2}\Rightarrow u=n'\frac{\pi}{2}$

$$= \frac{4}{\pi}\left[-\frac{1}{2n'^3}\underbrace{\int_0^{n'\frac{\pi}{2}} u^2\sin(u)\,du}_{"I_1} + \frac{\pi}{2n'^2}\underbrace{\int_0^{n'\frac{\pi}{2}} u\sin(u)\,du}_{"I_2}\ \underbrace{-\int_0^{\pi/2}\sin(n'x)\,dx}_{"I_3}\right]$$

$$I_1 = \int_0^{\pi/2} u^2\sin(u)\,du = -u^2\cos(u)\Big|_0^{n'\pi/2} + 2\cos(u)\Big|_0^{n'\pi/2} + 2x\sin(u)\Big|_0^{n'\frac{\pi}{2}}$$

$$= -\left(n'\frac{\pi}{2}\right)^2\cancelto{0}{\cos\left(n'\frac{\pi}{2}\right)} + 2\left(\cancelto{0}{\cos\left(n'\frac{\pi}{2}\right)} - \cancelto{1}{\cos(0)}\right) + 2\,\frac{n'\pi}{2}\cancelto{(-1)^n}{\sin\left(n'\frac{\pi}{2}\right)}$$

$$= -2 + n'\pi(-1)^n \quad / \; n'=2n+1$$

$$I_2 = \int_0^{\pi/2} u\sin(u)\,du = \sin(u)\Big|_0^{n'\pi/2} - u\cos(u)\Big|_0^{n'\pi/2}$$

$$= \sin\left(n'\frac{\pi}{2}\right) - \cancelto{0}{\sin(0)} - \left(n'\frac{\pi}{2}\cancelto{0}{\cos\left(n'\frac{\pi}{2}\right)} - \cancelto{0}{0\cdot\cos(0)}\right)$$

$$= (-1)^n$$

$$I_3 = -\int_0^{\pi/2}\sin(n'x) = \left[-\frac{1}{n'}\cos(n'x)\right]_0^{\pi/2} = \frac{1}{n'}\cancelto{0}{\cos\left(n'\frac{\pi}{2}\right)} - \frac{1}{n'}\cancelto{1}{\cos(0)}$$

Reemplazando:

$$A_{n'} = \frac{4}{\pi}\left[-\frac{1}{2n'^3}\left(-2+n'\pi(-1)^n\right)+\frac{\pi}{2n'^2}(-1)^n-\frac{1}{n'}\right]$$

$$= \frac{4}{\pi}\left[\frac{1}{n'^3}\cancel{-\frac{\pi}{2n'^2}(-1)^n}+\cancel{\frac{\pi}{2n'^2}(-1)^n}-\frac{1}{n'}\right]$$

$$= \frac{4}{\pi}\left[\frac{1}{n'^3}-\frac{1}{n'}\right]$$

$$= \frac{4}{\pi}\left[\frac{1}{(2n+1)^3}-\frac{1}{(2n+1)}\right]$$

e) La solución de $(*)$ $u(x,t)$ por lo tanto es:

$$u(x,t) = w(x,t) - f(x)$$

$$\boxed{= \sum_{n=0}^{\infty} \frac{4}{\pi}\left[\frac{1}{(2n+1)^3}-\frac{1}{(2n+1)}\right]\cos((2n+1)t)\sin((2n+1)x) - \frac{x^2}{2} + 1}$$

P_2

$$(P) = \begin{cases} \Delta u = 0 & \forall x\in\mathbb{R}\ \forall y>0 \quad (EQ) \\ u(x,0)=f(x) & \forall x\in\mathbb{R} \quad (CB) \\ \lim\limits_{y\to\infty}|u(x,y)|<\infty & \forall (x,y)\in\mathbb{R}^2 \end{cases}$$

a) Aplicando TF c/r a x con y fijo:

TF lineal

$$\widehat{u_{xx}+u_{yy}}\,(s,y)=0$$

$$\widehat{\frac{\partial^n}{\partial x^n}f(x)}\,(s)=(is)^n\hat{f}(s)$$

$$\widehat{u_{xx}}(s,y)+\widehat{u_{yy}}(s,y)=0$$

$$(is)^2\hat{u}(s,y)+\hat{u}_{yy}(s,y)=0$$

$$\Longrightarrow \boxed{\frac{d^2}{dy^2}\hat{u}(s,y) - s^2\hat{u}(s,y)=0}$$

EDO para variable y

Polinomio característico de la EDO (Ansatz $e^{\lambda s}$)

$$\lambda^2 - s^2 = 0$$

$$\Longrightarrow \lambda = \pm\sqrt{s^2}$$

$$= \pm|s|$$

$$\Rightarrow \underline{\hat{u}(s,y) = A(s)e^{|s|y} + B(s)e^{-|s|y}}$$

b) Imponiendo CB $u(x,0) = f(x)$ $\forall x \in \mathbb{R}$ y aplicando TF

$$\hat{u}(s,0) = f(x)$$

$$A(s)\underbrace{\cancel{e^{|s|0}}}_{1} + B(s)\underbrace{\cancel{e^{-|s|0}}}_{1} = \hat{f}(s)$$

$$\boxed{A(s) + B(s) = \hat{f}(s)}$$

Y como $\lim_{y\to\infty} |u(x,y)| < \infty \overset{\mathcal{F}}{\Rightarrow} \lim_{y\to\infty} |\hat{u}(s,y)| < \infty$ $\forall s \in \mathbb{R}$

$$\lim_{y\to\infty} \left| \underbrace{A(s)e^{|s|y}}_{\text{término diverge para } y\to\infty} + \underbrace{B(s)e^{-|s|y}}_{\text{se va a } 0} \right| < \infty \quad \forall s \in \mathbb{R}$$

Por lo que necesariamente $A(s) = 0$. Y por lo tanto:

$$B(s) = \hat{f}(s)$$

$$\Rightarrow \hat{u}(s,y) = \hat{f}(s)e^{-|s|y}$$

c) Para hallar $u(x,y)$ simplemente se antitransforma. Un tip útil para estos casos es utilizar el teorema de convolución:

$$\underline{\widehat{(f * g)(x)}(s) = \hat{f}(s)\,\hat{g}(s)}$$

Teo de convolución

Si $\hat{g}(s,y) = e^{-|s|y}$, entonces

$$\hat{u}(s,y) = \hat{f}(s)e^{-|s|y} = \widehat{(f*g)(x)}(s)\,\frac{1}{\sqrt{2\pi}}$$

$[\ ]^{\vee}(x)$ antiTF

$$u(x,y) = \frac{1}{\sqrt{2\pi}}(f*g)(x)$$

$$(f*g)(x)=\int_{-\infty}^{\infty} f(t)g(x-t)\,dt$$

definición convolución

$$=\frac{1}{\sqrt{2\pi}}\Big(f*\underbrace{[\hat{g}(s)]^{\vee}}_{=g}\Big)(x)$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,[\hat{g}(s)]^{\vee}(x-t)\,dt$$

Ahora solo necesitamos la antiTF de $\hat{g}(s,y)$:

$$g(x-t)=[\hat{g}(s,y)]^{\vee}(x-t)=[e^{-|s|y}]^{\vee}(x-t)$$

$$=\sqrt{\frac{2}{\pi}}\frac{y}{(x-t)^2+y^2}$$

ver aux 10 P1 (a) $(y>0)$

Reemplazando más arriba:

no depende de t

$$u(x,y)=\frac{1}{\sqrt{\cancel{2}\pi}}\int_{-\infty}^{\infty} f(t)\sqrt{\frac{\cancel{2}}{\pi}}\frac{y}{(x-t)^2+y^2}\,dt$$

$$=\frac{y}{\pi}\int_{-\infty}^{\infty} f(t)\frac{1}{(x-t)^2+y^2}\,dt$$