

P₁

$$f(x) = x^2, x \in [-\pi, \pi]$$

Encontrar la serie de Fourier de $f(x)$ implica encontrar los coefs a_n y b_n que hacen que

$$S_f = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \quad \begin{matrix} \nearrow 0 \text{ dado que} \\ f(x) \text{ par} \end{matrix}$$

Donde los coefs cumplen:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \quad n \in \{0, 1, 2, \dots\}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx) dx$$

$n=0$

$$\begin{aligned} \Rightarrow \frac{a_0}{2} &= \frac{2}{2\pi} \int_0^{\pi} x^2 dx \\ &= \frac{1}{\pi} \left. \frac{x^3}{3} \right|_0^{\pi} \\ &= \frac{\pi^2}{3} \end{aligned}$$

$n \neq 0$

$$a_n = \frac{1}{\pi} \int_0^{\pi} x^2 \cos(nx) dx$$

IPP (int. por partes)

$$u = x^2 \Rightarrow du = 2x dx$$

$$dv = \cos(nx) dx \Rightarrow v = \frac{1}{n} \sin(nx)$$

$$a_n = \frac{2}{\pi} \left(\left. \frac{x^2}{n} \sin(nx) \right|_0^{\pi} - \frac{2}{n} \int_0^{\pi} x \sin(nx) dx \right)$$

IPP de nuevo

$$u = x \Rightarrow du = dx$$

$$dv = \sin(nx) \Rightarrow v = -\frac{1}{n} \cos(nx)$$

$$\begin{aligned} \Rightarrow a_n &= \frac{1}{\pi} \left(\left. \frac{2x}{n^2} \cos(nx) \right|_0^{\pi} + \frac{2}{n} \left(-\frac{1}{n} \right) \int_0^{\pi} \cos(nx) dx \right) \\ &= \frac{1}{\pi} \left(\frac{2\pi}{n^2} \underbrace{\cos(n\pi)}_{(-1)^n} - \frac{2}{n^2} \left. \sin(nx) \right|_0^{\pi} \right) \end{aligned}$$

$$= \frac{1}{\cancel{\pi}} \cdot \frac{2\cancel{\pi}}{n^2} (-1)^n = \frac{2}{n^2} (-1)^n$$

Por lo que la serie es:

$$\underline{S_f = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos(nx)}$$

Y dado que x^2 es continua y $f(\pi) = f(-\pi)$, se tiene que $\forall x \in (-\pi, \pi)$ $S_f = f(x)$

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos(nx) \quad \times$$

b) Evaluando para $x = \pi$ *:

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \underbrace{(-1)^n}_{1} \underbrace{\cos(n\pi)}_1$$

$$\Rightarrow 4 \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2 - \frac{\pi^2}{3}$$

$$\underline{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4} \cdot \frac{2\pi^2}{3} = \frac{\pi^2}{6}}$$

P₂

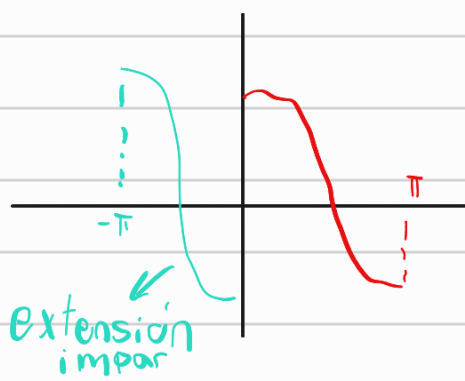
$$f(x) = \cos(x) \quad x \in [0, \pi]$$

$$\underline{\text{Pd q}} \quad f(x) = \sum_{n=1}^{\infty} \frac{8}{\pi} \frac{n}{4n^2 - 1} \sin(2nx)$$

Primero notamos que la expansión que se realiza es una serie en senos, las cuales se dan para expansiones de funciones impares. Por lo tanto extendemos $f(x)$ tq sea impar en $x \in [-\pi, \pi]$

$$\hat{f}(x) = \begin{cases} f(x) & x \in [0, \pi] \\ -f(x) & x \in [-\pi, 0] \end{cases}$$

$$\Rightarrow S_{\hat{f}} = \sum_{n=1}^{\infty} b_n \sin(nx)$$



donde los coefs b_n se calculan como

$\text{impar} \cdot \text{impar} = \text{par}$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \cos(x) \sin(nx) dx$$

$$= \frac{2}{\pi} \frac{1}{2} \int_0^{\pi} [\sin(nx-x) + \sin(nx+x)] dx$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} \sin(x(n-1)) dx + \int_0^{\pi} \sin(x(n+1)) dx \right]$$

$$= \frac{1}{\pi} \left[\frac{1}{n-1} (\cos(0) - \cos(\pi(n-1))) + \frac{1}{n+1} (\cos(0) - \cos(\pi(n+1))) \right]$$

$(-1) (-1)^{n-1} = (-1)^n$ $(-1) (-1)^{n+1} = (-1)^{n+2} = (-1)^n (-1)^2 = (-1)^n$

$$= \frac{1}{\pi} \left[\frac{1+(-1)^n}{n-1} + \frac{1+(-1)^n}{n+1} \right]$$

$$= \frac{1+(-1)^n}{\pi} \left[\frac{1}{n-1} + \frac{1}{n+1} \right]$$

$$= \begin{cases} 2 & n \text{ par} \\ 0 & n \text{ impar} \end{cases} \frac{2n}{n^2-1}$$

Sea $n' = 2n$, n par

$$\Rightarrow b_{2n=n'} = \frac{2 \cdot 2(2n)}{\pi} \frac{1}{4n^2-1}$$

$$= \frac{8}{\pi} \frac{n}{4n^2-1}$$

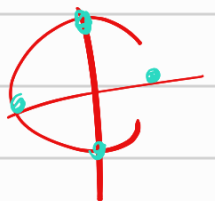
$$S_f = \sum_{n=1}^{\infty} \frac{8}{\pi} \frac{n}{4n^2-1} \sin(2nx) (*)$$

Ahora bien, para demostrar el valor de la serie

$$\frac{1}{2^2-1} - \frac{3}{6^2-1} + \frac{5}{10^2-1} - \frac{7}{14^2-1} + \dots$$

Evaluamos (*) en un pto adecuado. Si $x = \frac{\pi}{4}$

$$\frac{\sqrt{2}}{2} = \cos\left(\frac{\pi}{4}\right) = \sum_{n=1}^{\infty} \frac{8}{\pi} \frac{n}{4n^2-1} \sin\left(2n \frac{\pi}{4}\right)$$



$$\frac{\pi}{8} \left\{ \begin{array}{l} \frac{\sqrt{2}}{2} = \sum_{n=1}^{\infty} \frac{8}{\pi} \frac{n}{4n^2-1} \sin\left(\frac{n\pi}{2}\right) \\ n=2k-1 \end{array} \right. = \begin{cases} 0 & n \text{ par} \\ (-1)^{k+1} & n=2k-1 \text{ impar} \end{cases}$$

$$\frac{\pi\sqrt{2}}{16} = \sum_{k=1}^{\infty} \frac{2k-1}{4(2k-1)^2-1} (-1)^{k+1}$$

$$= \frac{1}{(2 \cdot 1)^2-1} - \frac{3}{(2 \cdot 3)^2-1} + \frac{5}{(2 \cdot 5)^2-1} - \frac{7}{(2 \cdot 7)^2-1} + \dots$$

P3

Sea $f: [-\pi, \pi] \rightarrow \mathbb{R}$ clase C^1 definida por:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

a) Pdg Identidad de Parseval:

$$\int_{-\pi}^{\pi} (f(x))^2 dx = \pi \frac{a_0^2}{2} + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

Partiendo del lado izq:

$$\int_{-\pi}^{\pi} (f(x))^2 dx = \int_{-\pi}^{\pi} \left[f(x) \cdot \underbrace{\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \right)}_{=f(x)} \right] dx$$

$f(\pi) = f(-\pi)$ y f continua ($\in C^1$)
 $\Rightarrow$ converge en media cuadrática y se puede intercambiar $\sum$ por $\int$

$$= \frac{a_0}{2} \int_{-\pi}^{\pi} f(x) dx + \int_{-\pi}^{\pi} f(x) \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) dx$$

$$= \pi \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(\int_{-\pi}^{\pi} f(x) (a_n \cos(nx) + b_n \sin(nx)) dx \right)$$

$$= \pi \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} f(x) \cos(nx) dx + b_n \int_{-\pi}^{\pi} f(x) \sin(nx) dx \right)$$

$$= \pi \frac{a_0^2}{2} + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$



b) Pdq $\int_{-\pi}^{\pi} (f'(x))^2 dx = \pi \sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2)$

Si $g(x) = f'(x)$:

$$\begin{aligned} g(x) &= \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \right)' \\ &= \sum_{n=1}^{\infty} (a_n (-\sin(nx)n) + b_n n \cos(nx)) \\ &= \sum_{n=1}^{\infty} (\underbrace{nb_n}_{a'_n} \cos(nx) + \underbrace{(-na_n)}_{b'_n} \sin(nx)) \end{aligned}$$

Como $f'(\pi) = f'(-\pi)$ y continua (clase C^1), entonces se puede aplicar Parseval para $g(x)$ con $a'_0 = 0$, $a'_n = nb_n$ y $b'_n = -na_n$:

$$\begin{aligned} \int_{-\pi}^{\pi} (f'(x))^2 dx &= \int_{-\pi}^{\pi} (g(x))^2 dx \\ &= \pi \frac{a'_0{}^2}{2} + \pi \sum_{n=1}^{\infty} (\underbrace{a_n'^2}_{nb_n^2} + \underbrace{b_n'^2}_{-na_n^2}) \quad \left. \begin{array}{l} \text{Parseval para} \\ g(x) \end{array} \right\} \\ &= \pi \sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2) \end{aligned}$$

