

BLUEPRINT FOR A PHYSICALLY-BASED, DIGITALLY-SIMULATED HYDROLOGIC RESPONSE MODEL

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Abstract: In recent years hydrologists have subjected the various subsystems of the hydrologic cycle to intensive study, designed to discover the mechanisms of flow and to arrive at physical and mathematical descriptions of the flow processes. As a consequence, meaningful results are now available in the form of numerical solutions to mathematical boundary value problems for groundwater flow, unsaturated porous media flow, overland flow, and channel flow. These developments in physical hydrology, together with the tremendous advance in digital computer technology, should provide the impetus for a necessary redirection of research in hydrologic simulation. In this paper, a blueprint for the development of physically-based hydrologic response models is presented; the level of sophistication that can be achieved with presently available methodology is discussed; and areas for necessary future research are pinpointed.

“The ability to accurately predict behavior is a severe test of the adequacy of knowledge in any subject.”

CRAWFORD and LINSLEY¹⁾

Introduction

“There is a group of hydrologists who espouse the pursuit of scientific research into the basic operation of each component of the hydrologic cycle in order to gain a full understanding of their mechanisms and interactions. Although the immediate motivation of an individual researcher may not transcend the narrow confines of a set of special phenomena, it is implicit that a full synthesis of the hydrologic cycle may eventually be sought. This concept of a full synthesis is held to be the only rational approach to hydrology.”

Amorocho and Hart²⁾

A complete physically-based synthesis of the hydrologic cycle is a concept

that tantalizes most hydrologists; and it is statements like the one above that give us hope and provide a *raison d'être* for papers such as ours. Unfortunately, this quote was taken somewhat out of context. Amorocho and Hart went on to say:

Progress in the area of physical hydrology has been considerable, but... it appears that neither the degree of knowledge gained to date nor the practical possibility of establishing accurate linkages between the component phenomena permits us to give full quantitative descriptions of natural hydrologic systems, except in very simple cases.

Crawford and Linsley¹⁾ had another objection:

Prohibitive amounts of input data would be required, far beyond practical limitations even for small experimental plots.

These reservations may still be valid, and it is not our purpose to make an emotional pitch for a pet methodology which can have no practical return. Rather, it is our purpose to examine the *possibility* of creating physically-based hydrologic response models; to review the level of sophistication that can be achieved with the available methodology; and to pinpoint the areas for necessary future research. We must concede that our paper is more of an "artist's conception" than a true "blueprint".

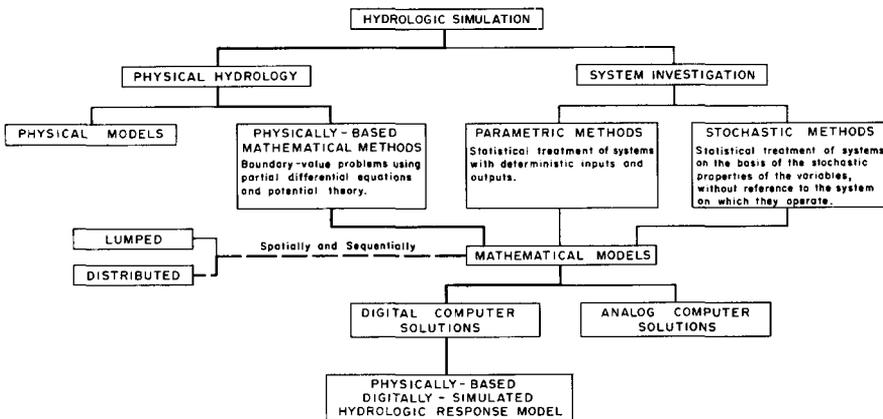


Fig. 1. Methods of hydrologic simulation (in part after Amorocho and Hart, 1964).

The purposes of a hydrologic response model are:

- (1) To synthesize past hydrologic events.
- (2) To predict future hydrologic events and to evaluate, for design purposes, combinations of hydrologic events occurring rarely in nature.
- (3) To evaluate the effects of artificial changes imposed by man on the hydrologic regime.

(4) To provide a means of research for improving our understanding of hydrology in general, and the runoff process in particular.

Figure 1 is a schematic diagram outlining the methods of hydrologic simulation. The two basic modes of approach can be classified into the broad categories of physical hydrology and hydrologic systems investigation.

Physical hydrology involves the systematic scientific investigation of the mechanisms of the component processes within the hydrologic cycle. If each of these processes can be described by a well-established physical law with an exact mathematical representation, then it should be possible to model entire watersheds. Such a model would be in the form of a composite boundary-value problem described by partial differential equations and potential theory.

The systems approach to hydrologic investigation incorporates the measurement of observable variables in the hydrologic cycle and the development of explicit relationships between these parameters. The two branches of system investigation are parametric and stochastic hydrology (Fig. 1).

Mathematical models of watershed hydrology can be derived from physically-based mathematical methods or by parametric or stochastic methods of system investigation. Perhaps the most important property of the mathematical model is the degree of representation given to the spatial and sequential variations in the input and output parameters. One can differentiate between a *lumped-system* model in which the watershed is treated as a "black box" and a *distributed-system* model in which the internal processes of the model are analyzed.

In Fig. 1, the heavy line represents our route through the methods of hydrologic simulation towards a physically-based, digitally-simulated hydrologic response model in which parameters are both spatially and sequentially distributed.

If we are to consider such a model, there are three sets of questions that must be answered; two are restatements of the reservations of Crawford and Linsley¹⁾ and Amorocho and Hart²⁾, and the third concerns computer technology.

(1) Are physically-based mathematical derivations of the hydrologic processes available? Are the interrelationships between the component phenomena well enough understood? Are the developments adaptable to a simulation of the entire hydrologic cycle?

(2) Is it possible to measure or estimate accurately the controlling hydrologic parameters? Are the amounts of necessary input data prohibitive?

(3) Have the earlier computer limitations of storage capacity and speed of computation been overcome? Is the application of digital computers to this type of problem economically feasible?

This paper deals almost exclusively with the first set of questions. We are convinced that in our own fields of specialization (infiltration, soil moisture, and groundwater flow) the answers are positive. We are less familiar with the overland flow and channel routing phases of the hydrologic cycle, but have arrived at some conclusions on the basis of the available literature. A point made by Dawdy and O'Donnell³⁾ is worth repeating in this context.

They noted that:

It is not necessary for the development of the over-all model approach that we wait until the complete specification of each of the elements of catchment behavior is completed. As the latter approach provides additional information, filling in the details of the picture, so will the over-all approach feed back information to show where further detailed specification is needed.

It is implicit that the interdependence between physical hydrology and system investigation could lead to the development of a hybrid approach to hydrologic simulation.

With regard to the second set of questions, the answers are more elusive. At present, the data available for most basins are insufficient for complete simulation. Even with more complete measurements, it will be necessary to extrapolate results of representative measurements of physical parameters to other points in the basin on the basis of a qualitative assessment of the geologic, pedologic, and meteorologic environments. A workable hydrologic response model will, therefore, be based in part on subjective considerations. *In situ* methods of measuring hydrologic parameters are still in the developmental stages, but it is our opinion that these methods are being developed and refined at a satisfactory rate, and that the development of physically-based hydrologic response models would give impetus to further progress.

As for computer capabilities; we offer a tentative answer in the form of a statement by Forsythe and Wasow⁴⁾:

The machines of the 1960's should permit time-dependant [boundary-value] problems in three dimensions to be attacked in moderate detail (i.e. with $100 \times 100 \times 100$ cubical nodal arrays).

What then of the machines of the 1970's?

In order to provide the reader with a comparative analysis, we will present in the following section a brief review of hydrologic systems investigation. We will then consider at greater length the development of physically-based, digitally-simulated hydrologic response models.

Hydrologic systems investigation

The hydrologic cycle is a dynamic system that operates within a set of

constraints or physical laws that control the movement, storage, and disposition of water within the system; the system derives its energy from the spatial imbalance between incoming and outgoing radiation. When considered with respect to the storage and movement of water within the system, the hydrologic cycle is a closed system and therefore, conforms to the principle of conservation of mass. Similarly, any realistic simulation model, although it may deal with only a segment of the hydrologic cycle, must maintain a balance between input, output, and storage, as well as meet certain fundamental requirements of "similitude" and generality:

(1) the model must simulate, on a continuous basis, the important processes and relationships within the system it represents,

(2) the model must be physically relevant to the system that it represents, and

(3) the model must be non-unique with respect to both time and space, and applicable over a wide range of hydrologic and geographic conditions.

The principal emphasis in systems investigation of hydrologic phenomena is on system operation which is dependant on the physical laws or constraints and the nature of the system (Dooge⁵). The nature of this dependance, however, need not be known and knowledge of the mechanisms of water transfer and storage phenomena within the hydrologic system in other than conceptual terms is not prerequisite to the development of a hydrologic simulation system.

The general systems simulation model consists of two principal components—storage elements and transmission routes—connected in parallel and in series by a set of *decision points*. Figure 2 represents a generalized conceptual model of such a hydrologic system. The operation of such a model involves the stepwise routing of precipitation or some other form of input through a "pot-and-pipe line" representation of the hydrologic cycle. The flow rate Q (Fig. 2, inset) is routed into storage elements A and B on the basis of pre-assigned routing criteria. Established routing criteria have been based upon empirical or statistical relationships which may or may not involve the flow rates Q , Q_A , Q_B , or the storage properties of elements A or B. In many cases these empirical or statistical relationships have physical relevance; whereas in others, they do not. In the overall routing procedure, each storage element in turn becomes a new decision point as the flow rate is routed through the model.

If, for example, Q is throughfall (Fig. 2), the decision point in question would represent the infiltration-overland flow decision point. Q_A represents the rate of infiltration, Q_B the overland flow rate, A soil-water storage, B overland-flow and channel storage, and Q_0 the rate of ground-water recharge. The rate of infiltration is commonly computed on the basis of threshold

concepts, infiltration curves, or exponential relationships. The excess is diverted to overland flow.

With hydrologic systems models, it is possible to simulate streamflow hydrographs with a high degree of accuracy for a variety of hydrologic and geographic conditions. The Stanford Watershed Model IV (Crawford and

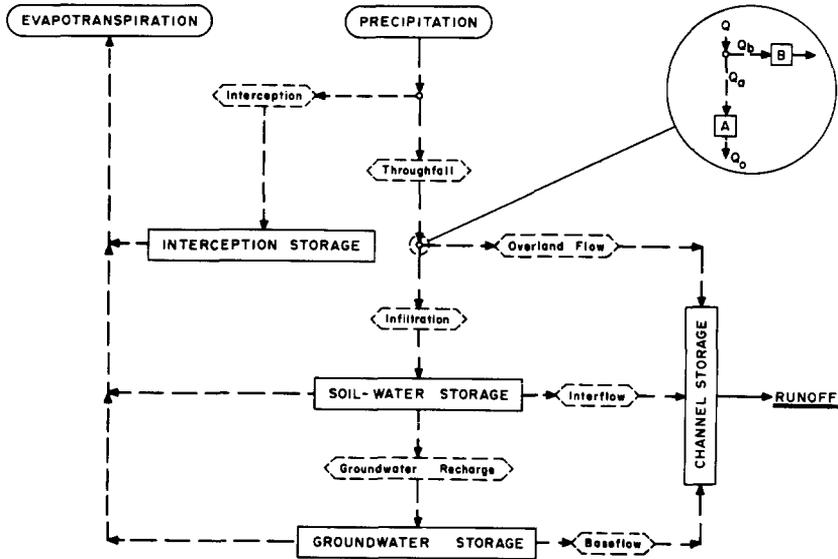


Fig. 2. A conceptual hydrologic model of the type used in the stepwise routing approach of systems hydrology.

Linsley¹) is the best-known and most successful model of this type. If the model we espouse is to offer promise for the future, it must be able to compete with the systems approach in terms of practical results and utility. A case could then be made for its superiority on the basis that a better understanding of the internal processes and their effects on the overall hydrologic system is desirable and could be beneficial to the solution of practical problems.

Physically-based, digitally-simulated hydrologic response model

CONCEPTUAL, THREE-DIMENSIONAL NODAL MODEL

In a physically-based mathematical model, the component, time-dependant hydrologic processes are represented by a set of partial differential equations, interrelated by the concepts of continuity of mass and of momentum. These

equations, together with the boundary conditions that define the shape and boundary properties of the basin, comprise the composite boundary value problem that is the hydrologic response model. A boundary value problem of this complexity must, by its very nature, be solved by numerical techniques and a digital computer. In numerical methods for solution of partial differential equations, the continuum of points making up the field and its boundaries is replaced by a finite set of points arranged in a grid over the region. Such a three-dimensional nodal grid system representing a hydrologic basin is illustrated schematically in Fig. 3.

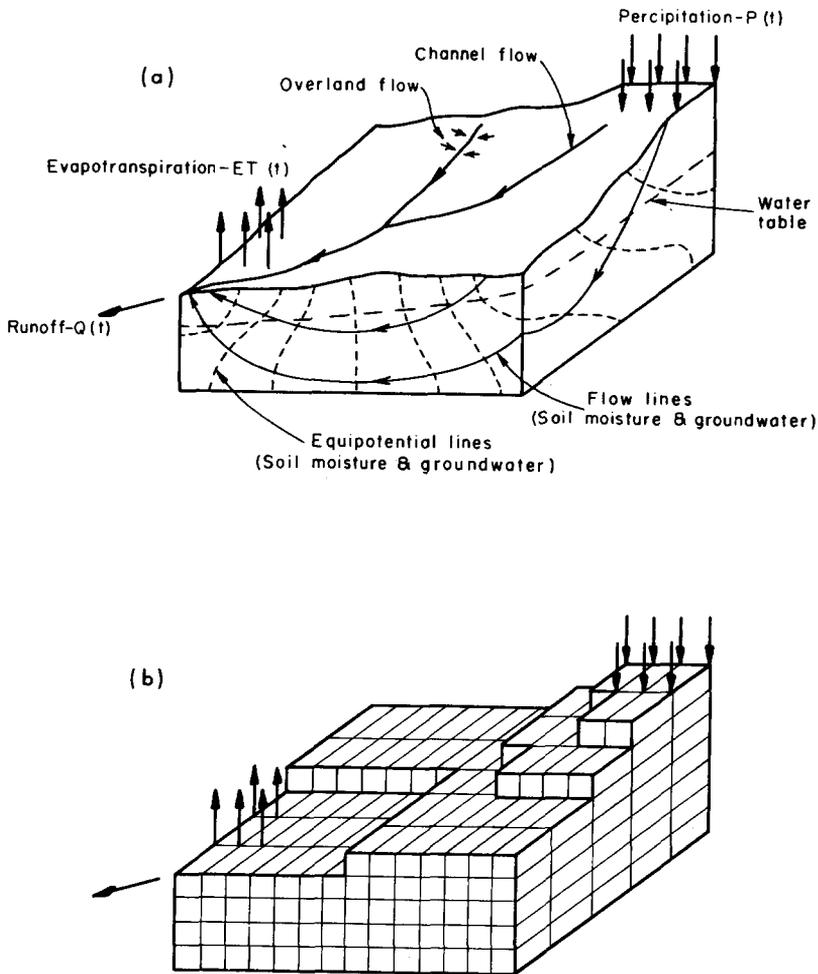


Fig. 3. Schematic diagram of (a) Hydrologic basin and (b) Three dimensional nodal model of hydrologic basin.

The operation of the physically-based, hydrologic response model requires basically four types of input:

(1) *Model definition input* specifies the size of the nodal grid system, the duration of the discrete time increments to be used to approximate the continuous timewise variations, the dimensions of the basin, and the topographic configuration. No restrictions are placed upon the configuration or size of the basin other than limitations imposed by computer capabilities.

(2) *Meteorological input* defines at each surface node the timewise variation in the flux of water arriving at or leaving the soil surface. No restrictions are made on the uniformity of the flux rate over the basin. In Fig. 3, we have shown a precipitation event and an evapo-transpiration event occurring simultaneously at different locations within the drainage basin.

(3) *Flow parameter input* specifies the control parameters for flow in the component hydrologic regimes. These control parameters include: Manning's n for overland and channel flow, hydraulic radii of stream channels, directional permeabilities in the ground-water zone, and hysteretic relationships between permeability, moisture content and moisture tension in the unsaturated soil-water zone. The use of a nodal grid system and numerical techniques permits consideration of non-homogeneity and anisotropy of flow parameters.

(4) *Mathematical input* in the form of the partial differential equations of the component hydrologic processes is not actually input to the model. These mathematical expressions, together with the boundary conditions, interface conditions, and decision processes, *are* the model, in the form of computer programs.

The output from the response model would provide a total picture of the hydrologic system. This output would include the streamflow hydrograph at any point within the basin, the groundwater flow pattern and the soil-moisture regime. The output would be continuous in both time and space and would incorporate surface-water, soil-water, and ground-water zones as components of a single system, not as discrete elements.

While the synthesis of the hydrologic cycle is continuous, the expressions describing the component phenomena have been developed independently by workers in many different fields. To assess the level of sophistication that is possible with the present methodology, it is still convenient to break the hydrologic regime into its time-honored blocks using the nomenclature of Fig. 2.

COMPONENTS OF THE MODEL

1. *Precipitation and evapotranspiration*

In the physically-based, mathematical model of catchment behavior,

meteorological input can be specified individually for each surface node. Both timewise and spatial variations can be incorporated. It is possible to simulate the hydrologic response of the watershed to both observed or hypothetical meteorological events.

The areal interpretation of pointwise precipitation measurements is a recognized hydrologic problem. The best available method appears to be the development of isohyetal patterns as outlined in most hydrologic texts. These patterns can then be used to determine input to the nodal system.

An area that presents a major difficulty in the development of a physically-based hydrologic response model is the determination of evapotranspiration which is highly variable over a drainage basin and with time. Much of the research in this area has been concerned with the development and testing of empirical or semi-empirical formulas for estimating potential evapotranspiration. Although several of the empirical forms give satisfactory estimates when used under the conditions for which they were developed, they do not permit simulation of evaporation and transpiration losses for successive, relatively short time increments, nor do they consider moisture availability. Pelton *et al.*⁶⁾ noted that where absolute values rather than comparative indexes are needed and where short-time values are important, the numerous empirical potential evapotranspiration methods are fundamentally and practically inadequate.

An encouraging recent development is the use of the three-dimensional partial differential equation of turbulent diffusion to determine evaporation from water surfaces. Brutsaert⁷⁾ has used this equation to derive the vertical vapor flux from a small water surface at ground level for the case in which eddy mixing is predominant and wind convection negligible. We feel that this type of approach holds the most promise for the eventual solution of the problem of evapotranspiration from soils.

Interception is another quantity that is difficult to calculate on a physical basis. At the present time interception remains an empirical quantity deducted from gross precipitation.

2. Infiltration and soil moisture flow

The physics of flow through porous media has been studied in great detail by groundwater hydrologists and soil physicists. The equation of flow, developed from Darcy's law and the equation of continuity, is:

$$\begin{aligned} \frac{\partial}{\partial x} \left[\rho K(x, y, z) \frac{\partial \phi}{\partial x} \right] + \frac{\partial}{\partial y} \left[\rho K(x, y, z) \frac{\partial \phi}{\partial y} \right] + \frac{\partial}{\partial z} \left[\rho K(x, y, z) \frac{\partial \phi}{\partial z} \right] \\ = \rho \frac{\partial \theta}{\partial t} + \theta \frac{\partial \rho}{\partial t} \quad (1) \end{aligned}$$

where:

x, y = horizontal coordinate directions
 z = vertical coordinate direction or
 elevation head (elevation above basal datum)

t = time

ρ = density of fluid

$K(x, y, z)$ = permeability of porous medium

φ = total hydraulic head

= $\psi + z$

$\psi \geq 0$ = pressure head (saturated)

$\psi < 0$ = soil moisture tension head (unsaturated)

θ = moisture content (unsaturated)

θ = porosity (saturated).

Equation (1) is a generalized equation of flow. From it, it is possible to develop the *one, two or three-dimensional* forms of the *steady or unsteady-flow* equations for *saturated or unsaturated* flow of a *compressible or incompressible* fluid through a *non-homogeneous, anisotropic* porous medium. The permeability K is spatially variable. In saturated flow this results solely from the inhomogeneity of the porous medium; in unsaturated flow it includes the effect of the variation in permeability with moisture tension, which is in turn a function of moisture content, and therefore, position.

For unsaturated flow, water is usually assumed to be incompressible, that is $\rho = \text{constant}$ and $\partial\rho/\partial t = 0$. Denoting $K(x, y, z)$ by the more usual $K(\psi)$, Eq. (1) reduces to:

$$\frac{\partial}{\partial x} \left[K(\psi) \frac{\partial \varphi}{\partial x} \right] + \frac{\partial}{\partial y} \left[K(\psi) \frac{\partial \varphi}{\partial y} \right] + \frac{\partial}{\partial z} \left[K(\psi) \frac{\partial \varphi}{\partial z} \right] = \frac{\partial \theta}{\partial t}. \quad (2)$$

Defining the specific moisture capacity, C , as $C(\psi) = \partial\theta/\partial\psi$, and reducing to one-dimensional vertical form, Eq. (2) becomes:

$$\frac{\partial}{\partial z} \left[K(\psi) \left(\frac{\partial \psi}{\partial z} + 1 \right) \right] = C(\psi) \frac{\partial \psi}{\partial t}. \quad (3)$$

This equation for one-dimensional vertical soil-moisture flow, or its equivalent, has been solved by many authors for a wide variety of boundary conditions and soil properties. Significant contributions were made by Klute⁸⁾ who first used the finite-difference approach, by Philip⁹⁾ in his monumental 7-part paper on infiltration, by Hanks and Bowers¹⁰⁾ for layered soils, by Rubin and Steinhardt¹¹⁾ for the constant-rate rainfall case, by Whisler and Klute¹²⁾ for the consideration of hysteresis in the functional relationships between K , ψ , C and θ , and by Staple¹³⁾ and Rubin¹⁴⁾ for the redistribution problem.

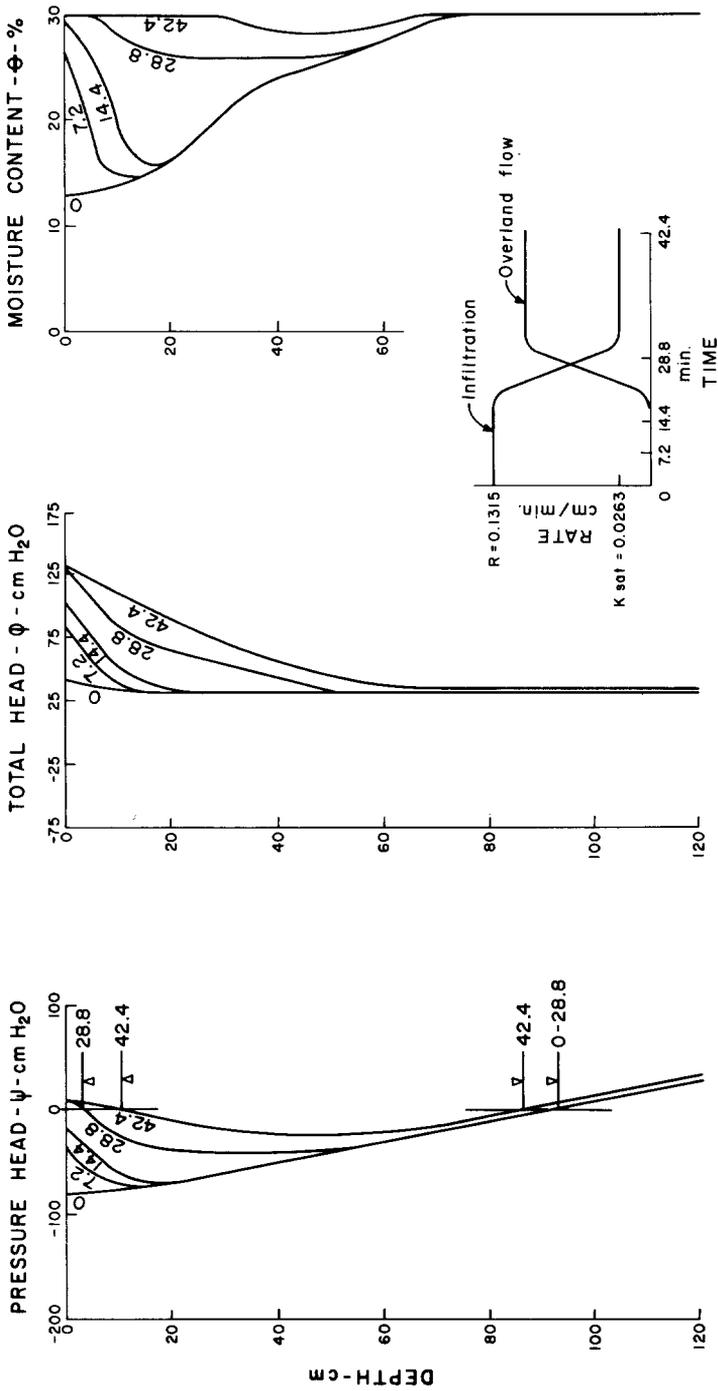


Fig. 4. Numerical solution to a mathematical model of one-dimensional vertical infiltration into a recharging groundwater flow system. $Q = 0.001315$ cm/min; $R = 0.1315$ cm/min; Maximum surface head = 10 cm; Soil = Del Monte Sand.

The continuity of flow between the saturated and unsaturated zones has recently been studied by Freeze¹⁵). Results from this study serve to illustrate the applicability of the method to hydrologic response modeling. Figure 4 shows the pressure-head, total-head and moisture-content profiles for a case of one-dimensional vertical infiltration into a recharging groundwater flow system. The pressure-head profile is the result obtained from the finite-difference solution to Eq. (3) under the boundary conditions of a constant rainfall rate R at the surface and a constant rate of groundwater recharge Q out the base. For this case the rainfall rate was 100 times the groundwater recharge rate and 5 times the saturated permeability of the soil. Rainfall at a rate of this intensity is seldom measured but it was chosen because it serves best to illustrate the principles. The initial conditions, as indicated by the profiles labelled zero, represent steady state downward flow at the rate Q . The initial depth to the water table ($\psi = 0$) was 92 cm. Subsequent profiles are labelled with the number of minutes elapsed since the beginning of the run.

An interpretation of the pressure-head profile in Fig. 4 shows that this rainfall intensity and these antecedent conditions caused saturation ($\psi > 0$) at the surface about 20 minutes after the start of the rainfall. Pondered water then resulted until the maximum depth of ponding (10 cm in this case) was reached. During this time, a saturated layer of increasing thickness developed at the surface. Between times 28.8 and 42.4 min, the water table rose from a depth of 92 cm to 86 cm.

One can interpret the total-head profile in the center of Fig. 4 in terms of the direction and magnitude of the hydraulic gradient throughout the soil column. The moisture-content profiles are self-explanatory. After 42.4 min of infiltration the column was saturated over almost its full length.

It is possible to calculate the flux into the soil column at the surface. It will vary with time depending on the surface hydraulic gradients and the permeability values. The quantity of water that arrives at the surface as precipitation but does not infiltrate into the soil column is available for overland flow. The inset in Fig. 4 is a plot of the rates of infiltration and overland flow vs time for this particular case.

If we assume this one-dimensional treatment to be taking place below a given surface node in Fig. 3, its place in the integrated hydrologic model is clear. Routing of the water at the "decision point" at the ground surface into either infiltration or overland flow is thus carried out by a mathematical model based on the physical process. There is no need to make use of empirical infiltration curves, the tenuous concept of infiltration capacity, outdated parameters such as wilting point or field capacity, or questionable results from infiltrometer tests. What *are* needed are measurable parameters of physical significance, namely the initial soil-moisture conditions, and the relationship

between permeability, specific moisture capacity, moisture content and soil-moisture tension for the soil in question. Laboratory measurements of these latter variables are available for some soils and could be measured as standard hydrologic practice for others. In situ measurement techniques are not yet available.

The same method but with different boundary conditions can be used for cases involving evapotranspiration from the surface, or inflowing groundwater discharge at the base. For the evapotranspiration case, research is needed to investigate the relationship between meteorological conditions and the resulting surface flux.

Rubin¹⁶⁾ has recently pioneered the development of numerical solutions for transient flow of water in *two-dimensional* saturated-unsaturated systems.

Developmental work on three-dimensional systems is still in progress. The impending breakthrough will allow the calculation of three-dimensional soil-moisture flow systems similar to and contiguous with the groundwater-flow systems which will be described in the following section. Quantitative interpretation of these unsaturated-flow systems will provide a method of analyzing the interflow component of surface runoff.

3. *Groundwater flow*

Freeze and Witherspoon¹⁷⁾, building on the groundwork laid by Toth¹⁸⁾, showed that it is possible to simulate steady-state regional groundwater-flow patterns in a three-dimensional, non-homogeneous, anisotropic groundwater basin by obtaining finite-difference solutions to an appropriate mathematical model. The partial differential equation that describes the flow is Eq. (1) with the right-hand side equal to zero. The boundary conditions are based on the assumption that the groundwater basin is bounded on the bottom by a horizontal impermeable basement, on the top by the water-table and on all sides by imaginary vertical impermeable boundaries which simulate the groundwater divides.

Figure 5 shows a two-dimensional flow pattern (after Freeze and Witherspoon¹⁹⁾) through the Gravelbourg aquifer in Saskatchewan, Canada, as determined by the numerical solution to a mathematical model. The diagram shows four geological formations and the measured or estimated values of horizontal and vertical permeability for each. The water-table configuration is based on well records. The equipotential lines are shown and the directions of groundwater flow are indicated by arrows. The results are in good agreement with field measurements.

It is possible to analyze these mathematically-derived groundwater-flow systems quantitatively in order to calculate the natural basin yield and to determine the rate of recharge (flow away from the water table within the

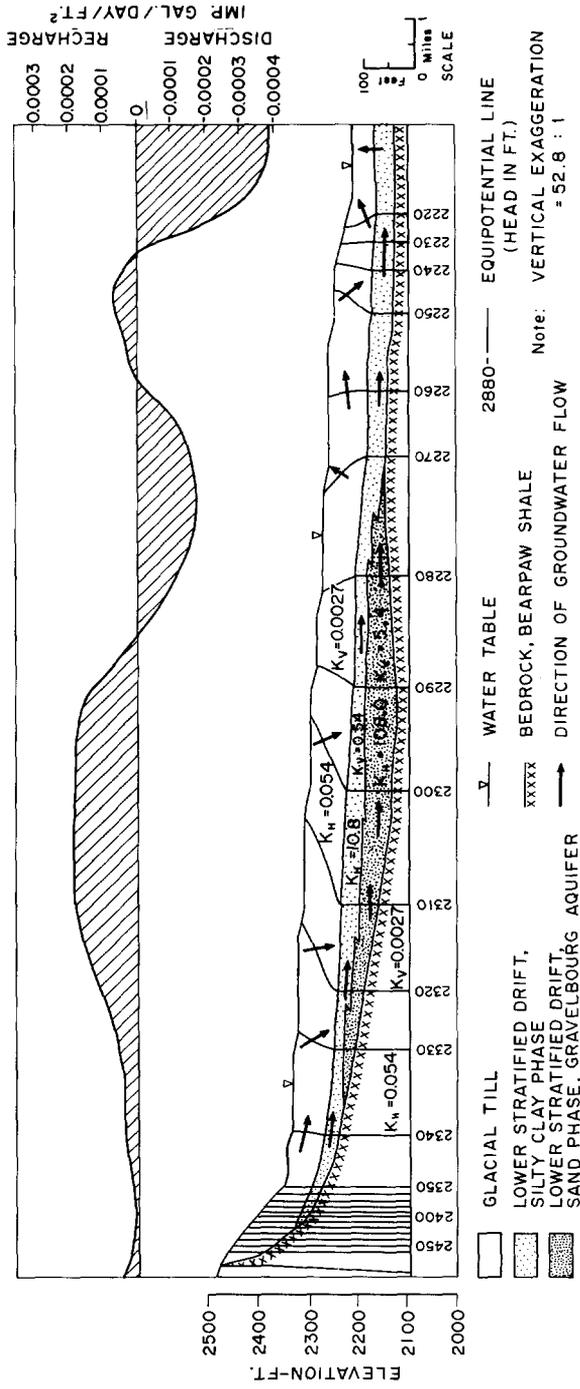


Fig. 5. Two-dimensional flow pattern through the Gravelbourg aquifer, Saskatchewan, Canada as determined by a numerical solution to a mathematical model.

saturated zone) at any given point on the water table. The plot above the flow pattern in Fig. 5 is a "recharge-discharge profile" which graphically shows these rates. The rate of flux across the water table varies widely across the basin.

The rate of recharge or discharge determined by a quantitative evaluation of the steady-state regional groundwater-flow pattern is the quantity used as the basal boundary condition in the one-dimensional vertical soil-moisture flow regimes presented in the previous section and in Fig. 4.

In order to simulate the groundwater component of a hydrologic response model, it should be sufficient to use an average water-table position in a steady state analysis of regional groundwater flow, if:

- (1) the zone of fluctuation of the water table is only a small percentage of the total saturated depth of the groundwater basin, and
- (2) the relative configuration of the water table remains the same.

If these two conditions are not satisfied, or if there is major well-field development within the basin, then a transient mathematical model will be necessary. Two-dimensional horizontal transient mathematical models using numerical finite-difference solutions have been developed for aquifer analysis by Tyson and Weber²⁰) and Bittinger, Duke and Longenbaugh²¹). Pinder (personal communication) has developed a two-dimensional transient model with vertical leakage.

Transient models in three dimensions, valid for the entire groundwater zone, have not yet been developed, although the partial differential equation is known (De Wiest²²):

$$\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}\right) + 2\rho g \beta \left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2 - \frac{\partial \varphi}{\partial z} \right] = \frac{\rho g}{k} [(1 - \theta) \alpha + \theta \beta] \frac{\partial \varphi}{\partial t}. \quad (4)$$

This equation can be developed from Eq. (1). The additional symbols are:

- g = acceleration due to gravity
- α = vertical compressibility of the granular skeleton of the porous medium
- β = compressibility of the fluid.

There may be problems in specifying the compressibility terms so as to maintain continuity between confined and unconfined aquifers.

The finite-element method is a powerful new approach for handling problems of transient fluid flow in complex systems (Javandel and Witherspoon²³). It may well be that the generality of this approach will make it the ultimate tool in the design of mathematical hydrologic response models.

4. Overland and channel flow

Overland flow, characterized as unsteady, spatially-varied sheetflow, is defined in classical hydrology as the precipitation excess that moves over the land surface to stream channels after infiltration. It is that part of the total surface runoff not confined to the stream channels. Except in agriculturally smooth or highly urbanized areas, however, overland flow in the classical sense is rarely observed in nature. Rather it occurs through small conveyances and therefore, approaches channel flow. This close association between channel flow and overland flow is further borne out by the closely parallel development of techniques to study and analyze each type of flow.

Both overland and channel flow can be described by the one-dimensional hydrodynamic equations of continuity and momentum for unsteady, non-uniform, spatially-varied open-channel flow:

$$y \frac{\partial v}{\partial x} + v \frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} = q - i \quad (5)$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + g \frac{\partial y}{\partial x} = g(S_0 - S_f) - \frac{v}{y}(q - i) \quad (6)$$

where:

g = acceleration due to gravity

q = channel inflow rate

i = channel infiltration rate

t = time

x = coordinate direction

y = flow depth

v = flow velocity

S_0 = channel bottom slope

S_f = friction slope

where:

$$S_f = \frac{v^2 n^2}{2.2082 R^{\frac{4}{3}}} \quad (7)$$

and

n = Manning's roughness coefficient

R = channel hydraulic radius.

Although the partial differential Eqs. (5) and (6) which are attributable to St. Venant, have been known since the 19th century, it was not until the advancement of numerical techniques and the digital computer that their solution in complete form became possible. It appears that these equations were first solved mathematically as part of a mathematical hydrologic model by Isaacson *et al.*²⁴). Well-documented finite-difference solutions have

since been presented by Morgali and Linsley²⁵) and Brakensiek²⁶). A recent paper by Liggett and Woolhiser²⁷) summarizes and compares available numerical methods for solution of the hydrodynamic equations.

Figure 6 is a schematic diagram of a mathematical model of surface

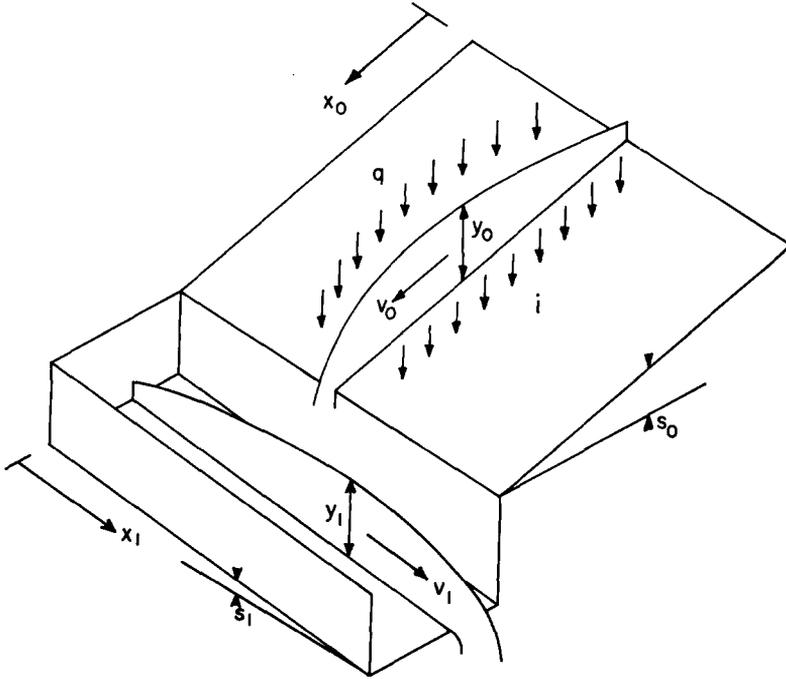


Fig. 6 Schematic diagram of an overland flow plane discharging into a channel (after Liggett and Woolhiser, 1967).

runoff showing an overland flow plane discharging into a channel. Inflows at the upper boundary of the overland flow reach ($x_0=0$) and channel reach ($x_1=0$) are independent and may be constant, variable, or zero. The downstream boundary conditions may be specified in terms of discharge or may represent free outflow or pool conditions. The inflow rate q , to the overland flow plane, results from rainfall. Inflow to the channel is the result of lateral inflows including overland flow, baseflow, and interflow. The basal boundary may be impervious, or pervious with a channel infiltration rate i . The initial condition for overland flow is usually a dry channel, whereas for channel flow the initial conditions are defined by the rates of groundwater discharge and interflow.

The output from such a mathematical model of overland and channel flow is the timewise variation in depth, velocity, and discharge at each control

section. Morgali and Linsley²⁵) have solved the hydrodynamic equations for the specific case of an impervious channel; Kruger and Bassett²⁸) considered the effects of channel infiltration, and Ragan²⁹) the presence and effects of lateral channel inflows in the solution of the differential equations.

Several authors have attempted to integrate overland flow and channel-routing procedures into a total runoff model. Wooding³⁰) did so by solving the kinematic wave equations, a simplified form of the hydrodynamic equations, for both catchment and stream. Harbaugh and Chow³¹) developed a numerical mathematical model, based on the complete form of the hydrodynamic equations, in which the overland flow is analyzed first with rainfall as the spatially-varied inflow; channel flow is analyzed, with the computed overland flow acting as spatially-varied inflow. Machmeier and Larson³²) routed unsteady flow through an idealized channel system using a finite-difference solution to the complete hydrodynamic equations.

The areally-variable flow parameter prerequisite to the operation of a mathematical model is Mannings n as defined by Eq. (7) for flow between surface nodes. Although Mannings n was developed originally for steady flow, it has been successfully applied to unsteady, non-uniform, turbulent flow. Mannings n is not constant at any given point, but varies with time due to changes in channel slope, channel dimensions, depth of flow, vegetal cover, and configuration of the channel bottom. Ragan²⁹) noted that his numerical solutions were sensitive to errors in the roughness coefficient and to minimize this sensitivity, he recommended further investigations directed toward adaptation of numerical techniques to field conditions.

In the three-dimensional, nodal, hydrologic response model (Fig. 3), the method of routing both overland and channel flow is dependant upon the surface topographic configuration and the position of defined flow channels. Infiltration from the channel bed is dependant on the position of the channel with respect to the recharge-discharge regime of the saturated-unsaturated subsurface flow system (Fig. 5). In some cases groundwater discharge will be a source of lateral inflow to overland flow, whereas in others overland flow will become a primary source of infiltration.

Liggett and Woolhiser³³) have some serious reservations regarding the possible use of the shallow water equations in mathematical models of surface runoff. Their conclusions can be summarized as follows:

(1) The runoff process is extremely complex and cannot be described completely in a mathematical way. Direct use of the shallow water equations will not lead to significant improvements in runoff models. However, solutions to the shallow water equations provide insight into the physical process and aid in evaluating less complex models. The question is: What components of this phase of the hydrologic system need be considered, and

can they be treated in such a manner that physical significance is retained while a suitable degree of simplification is provided.

(2) For most overland flow problems, the kinematic wave equations should prove as suitable as the complete hydrodynamic equations, and they are simpler to solve.

(3) Simple parametric relationships between velocity and depth may prove more useful than attempting to apply Eq. (7), with the attendant problem of measuring n and R .

In the surface runoff phase, then, it appears that our desire for a physically-based model must be tempered with the practical considerations which point toward parametric analysis of some of the components. If such a hybrid approach proves successful, the eventual development of mathematical models that can be integrated into a composite physically-based synthesis seems assured.

Conclusions

Our purpose has been to assess the feasibility of the development of a "rigorous", physically-based mathematical model of the complete hydrologic system. Many physically-based mathematical derivations of hydrologic processes are now available. The present level of sophistication allows treatment of:

(1) one- and two-dimensional transient soil-moisture flow in non-homogeneous soils; three-dimensional treatments are imminent.

(2) three-dimensional, steady-state groundwater flow in non-homogeneous, anisotropic formations; two-dimensional, horizontal, transient groundwater flow in homogeneous, isotropic, confined aquifers.

(3) one-dimensional, unsteady, non-uniform, spatially-varied open-channel flow, with lateral inflow and channel infiltration.

The level of development is not adequate to permit the construction of complete physically-based, hydrologic response models at this time. The recent progress in physical hydrology should, however, encourage continued research directed toward the eventual establishment of such models. These models will take the form of three-dimensional mathematical boundary-value problems with spatially and sequentially distributed inputs, solved by numerical methods with the aid of digital computers.

Further research is needed to define:

(1) the physical relationship between meteorological phenomena and evapotranspiration from unsaturated soil.

(2) the continuity between saturated and unsaturated flow in two- and three-dimensional, non-homogeneous, anisotropic systems.

(3) the continuity between the mathematical developments for ground-water flow in confined and unconfined aquifers.

(4) channel flow in irregular natural channels under non-steady state conditions.

(5) the role of vegetation in the hydrologic flow system.

It should not be necessary to await complete synthesis of the hydrologic cycle in order to make use of recent advancements in physical hydrology. For example, the numerical mathematical model of soil-moisture flow used by soil scientists and reviewed in this paper provides a more rational approach to the determination of infiltration than does the use of parameters such as infiltration capacity, wilting point, and field capacity.

In recognition of the quantity of input data necessitated by the physically-based approach, simplification of the model is needed to reduce the complete model to workable dimensions while maintaining physical relevance of the controlling hydrologic parameters. For example the analysis of the surface runoff component may best be handled with a simplified form of the shallow water equations, using parametric relationships where necessary. Such a simplification procedure would, by retaining physical significance in the controlling parameters, permit us to extrapolate rather than merely interpolate, results.

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