P1 Resuelva el sgte PVI

$$y'' + 9y = 2\,\text{sen}(3t)$$

$$\begin{cases} y(0) = 1 \\ y'(0) = 0. \end{cases}$$

Sol: Tomando transf. de Laplace, se obtiene que

$$(s^2+9)\mathcal{L}(y)(s) - s\cdot \overset{1}{\cancel{y(0)}} - \overset{0}{\cancel{y'(0)}} = 2\,\frac{3}{s^2+9}$$

$$\Rightarrow (s^2+9)\mathcal{L}(y)(s) - s = \frac{6}{s^2+9}$$

$$\Rightarrow \mathcal{L}(y)(s) = \frac{6}{(s^2+9)^2} + \boxed{\frac{s}{s^2+9}}$$

$\hookrightarrow \mathcal{L}(\cos(3t))(s)$

Además,

$$\frac{6}{(s^2+9)^2} = \frac{2}{3}\,\frac{3}{(s^2+9)}\cdot\frac{3}{(s^2+9)}$$

$$= \frac{2}{3}\mathcal{L}(\text{sen}(3t))(s)\cdot\mathcal{L}(\text{sen}(3t))(s)$$

ya que $= \frac{2}{3}\mathcal{L}(\text{sen}(3t) * \text{sen}(3t))(s)$

$$\frac{2}{3}\,\text{sen}(3t) * \text{sen}(3t) = \frac{2}{3}\int_0^t \text{sen}(3u)\cdot\text{sen}(3(t-u))\,du$$

②

∏ Recordemos que

$$\cos(A+B) = \cos(A)\cos(B) - \operatorname{sen}(A)\operatorname{sen}(B)$$

$$\cos(A-B) = \cos(A)\cos(B) + \operatorname{sen}(A)\operatorname{sen}(B)$$

$$\Rightarrow \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right) = \operatorname{sen}(A)\operatorname{sen}(B).$$

$$\Rightarrow \frac{2}{3}\operatorname{sen}(3t) * \operatorname{sen}(3t) = \frac{2}{3}\cdot\frac{1}{2}\int_0^t \cos(3u - 3(t-u)) - \cos(3u + 3(t-u))\, du$$

$$= \frac{1}{3}\int_0^t \cos(6u - 3t) - \cos(3t)\, du$$

$$= \frac{1}{3}\left.\frac{\operatorname{sen}(6u-3t)}{6}\right|_0^t - \frac{1}{3}t\cos(3t)$$

$$= \frac{1}{18}\left(\operatorname{sen}(3t) + \operatorname{sen}(3t)\right) - \frac{1}{3}t\cos(3t)$$

$$= \frac{1}{9}\operatorname{sen}(3t) - \frac{1}{3}t\cos(3t)$$

Entonces, tenemos

$$\mathcal{L}(y)(s) = \mathcal{L}\left(\frac{1}{9}\operatorname{sen}(3t) - \frac{1}{3}t\cos(3t)\right)(s) + \mathcal{L}(\cos(3t))(s)$$

$$\Rightarrow y(t) = -\frac{1}{3}t\cos(3t) + \frac{1}{9}\operatorname{sen}(3t) + \cos(3t).$$

□.

P2) Se sabe que

$$t y'' + y' + t y = 0$$

tiene solución $y:[0,\infty) \to \mathbb{R}$ con y'', y' de orden exponencial y que cumple $y(0)=1$ | $y'(0)=0$.

a) Demuestre que si $Y(s) = \mathcal{L}(y)(s)$, entonces

$$(1+s^2)\, Y'(s) + s\, Y(s) = 0.$$

Dem: Recordando la propiedad

$$\mathcal{L}(t f(t)) = -\frac{dF}{ds}$$

vemos que

$$\mathcal{L}(t y'') + s Y(s) - \overset{1}{y(0)} + \mathcal{L}(t y)(s) = 0$$

$$\Rightarrow -\frac{d}{ds}\left(\mathcal{L}(y'')(s)\right) + s Y(s) - 1 - Y'(s) = 0$$

$$\Rightarrow -\frac{d}{ds}\left(s^2 Y(s) - s\overset{1}{y(0)} - \overset{0}{y'(0)}\right) + s Y(s) - 1 - Y'(s) = 0$$

$$\Rightarrow -\left(2s Y(s) + s^2 Y'(s) - 1\right) + s Y(s) - 1 - Y'(s) = 0$$

$$\Rightarrow -2s Y(s) + s^2 Y'(s) + \cancel{1} + \cancel{s Y(s)} - \cancel{1} + Y'(s) = 0$$

$$+ s Y(s)$$

$$\Rightarrow (s^2+1)\, Y'(s) + s Y(s) = 0. \quad \square$$

b) Demuestre que
$$\mathcal{L}(y)(s) = \frac{1}{\sqrt{1+s^2}}.$$

Dem: sabemos que
$$(s^2+1)\,Y'(s) + s\,Y(s) = 0$$
$$\Leftrightarrow \quad Y'(s) + \frac{s}{s^2+1} Y(s) = 0.$$

Por lo que podemos obtener $Y(s)$ como solución de esta EDO. Notamos que
$$\int \frac{s}{s^2+1}\,ds = \frac{1}{2}\int \frac{2s}{s^2+1}\,ds = \frac{1}{2}\ln(s^2+1)$$
$$\Rightarrow e^{\int \frac{s}{s^2+1}ds} = e^{\frac{1}{2}\ln(s^2+1)} = e^{\ln\sqrt{s^2+1}} = \sqrt{s^2+1}$$

$$\Rightarrow \sqrt{s^2+1}\,Y'(s) + Y(s)\frac{s}{\sqrt{s^2+1}} = 0$$

$$\Leftrightarrow \left(\sqrt{s^2+1}\,Y(s)\right)' = 0 \quad \Rightarrow \sqrt{s^2+1}\,Y(s) = C$$

$$\Rightarrow Y(s) = \frac{C}{\sqrt{s^2+1}}$$ (Debemos determinar la cte de integración)

Sabemos lo siguiente sobre la transformada de Laplace:
$$f(0) = \lim_{s\to\infty} sF(s)$$

Entonces como $y(0)=1$, se tendría que tener
$$\lim_{s\to\infty} s\frac{C}{\sqrt{s^2+1}} = 1 \Rightarrow \boxed{C=1}. \quad \square$$

P3) (Laplace para sistemas) 5

encuentre u, v tales que

$$(1) \quad \frac{du}{dx} + u + \frac{dv}{dx} + v = 1 \qquad u(0) = 1$$

$$(2) \quad 2\frac{du}{dx} + u + \frac{dv}{dx} = 0 \qquad v(0) = 0$$

Sol: Tomando Laplace, obtenemos

$$(1) \quad sU(s) - \overset{1}{u(0)} + U(s) + sV(s) - \overset{0}{v(0)} + V(s) = \frac{1}{s}$$

$$(2) \quad 2\left(sU(s) - \overset{1}{u(0)}\right) + U(s) + sV(s) - \overset{0}{v(0)} = 0$$

$$\Rightarrow \begin{cases} (1+s)U(s) + (1+s)V(s) = \frac{1}{s} + 1 = \left(\frac{1+s}{s}\right) \\ (1+2s)U(s) + sV(s) = 2 \end{cases}$$

$$\Rightarrow \begin{pmatrix} 1+s & 1+s \\ 1+2s & s \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 1+s/s \\ 2 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} U \\ V \end{pmatrix} = \frac{1}{s(1+s) - (1+s)(1+2s)} \begin{pmatrix} s & -(1+s) \\ -(1+2s) & 1+s \end{pmatrix} \begin{pmatrix} \frac{1+s}{s} \\ 2 \end{pmatrix}$$

$$= \frac{-1}{(1+s)^2} \begin{pmatrix} 1+s & -2(1+s) \\ -\frac{(1+2s)(1+s)}{s} & +2(1+s) \end{pmatrix} = \frac{-1}{(1+s)^2} \begin{pmatrix} -(1+s) \\ '' \end{pmatrix}$$

$$= \begin{pmatrix} 1/1+s \\ \frac{-2}{1+s} + \frac{(1+2s)}{s(1+s)} \end{pmatrix} \rightarrow = \frac{1}{(1+s)}\left(-2 + \frac{1+2s}{s}\right) = \frac{1}{1+s}\left(\frac{-2s+1+2s}{s}\right)$$

$$= \frac{1}{1+s} \cdot \frac{1}{s} = \frac{1}{s} - \frac{1}{1+s}$$

obtuvimos entonces

(1) $U(s) = \frac{1}{1+s} \Rightarrow u(t) = e^{-t}$

(2) $V(s) = \frac{1}{s} - \frac{1}{1+s} \Rightarrow v(t) = 1 - e^{-t}$