P_1

$$(*) = \begin{cases} u_{tt} = u_{xx} + 1, \; x \in (0, \pi/2), t > 0 & \text{(EQ)} \\ u(0,t) = 1, \quad u_x(\pi/2, t) = -\pi/2 & \text{(CB)} \\ u(x,0) = -x^2 + \frac{1}{2}\pi x, \; u_t(x,0) = 0, \; x \in (0, \pi/2) & \text{(CI)} \end{cases}$$

a) Consideramos el cambio de variables $w(x,t) = u(x,t) + f(x)$ $\Rightarrow u(x,t) = w(x,t) - f(x)$. Como sabemos que u es solución de $(*)$, reemplazando en (EQ):

$$u_{tt} = u_{xx} + 1 \overset{u = w - f}{\Longrightarrow} w_{tt} - \cancel{f_{tt}} = w_{xx} - \underbrace{f_{xx}}_{f''(x)} + 1$$

0 f no depende de t

Queremos llegar a una (EQ') igual a $w_{tt} = w_{xx}$, por lo tanto:

$$w_{tt} = w_{xx} \underbrace{- f''(x) + 1}_{\text{queremos que sea 0 para llegar a lo que queremos}}$$

desde integar

$$\Rightarrow f''(x) = 1$$

$\int dx + c_1$

$$f'(x) = x + c_1$$

$\int dx + c_2$

$$f(x) = \frac{x^2}{2} + c_1 x + c_2$$

Reemplazando $u = w - f$ en (CB):

$$u(0,t) = 1 \Rightarrow w(0,t) - f(0) = 1$$

$$w(0,t) = \underbrace{1 + f(0)}_{=0 \text{ por lo que se quiere llegar}}$$

$$\Rightarrow f(0) = -1$$

$$u_x(\pi/2,t) = -\pi/2 \Rightarrow w_x(\pi/2,t) - f'(\pi/2) = -\pi/2$$

$$w_x(\pi/2,t) = \underbrace{f'(\pi/2) - \pi/2}_{=0}$$

$$\Rightarrow f'(\pi/2) = \pi/2$$

Evaluando f en los ptos que encontramos:

$$f(x) = \frac{x^2}{2} + c_1 x + c_2 \wedge f'(x) = x + c_1$$

$$f(0) = -1 \Rightarrow \boxed{c_2 = -1}$$

$$f'(\pi/2) = \pi/2 \Rightarrow \cancel{\pi/2} + c_1 = \cancel{\pi/2}$$

$$\Rightarrow \boxed{c_1 = 0}$$

Por último falta comprobar que $f(x) = \frac{x^2}{2} - 1$ comprueba las (CI') para $w(x,t)$:

$$u(x,0) = -x^2 + \frac{1}{2}\pi x \Rightarrow w(x,0) - f(x) = -x^2 + \frac{1}{2}\pi x$$

$$\Rightarrow w(x,0) = -x^2 + \frac{\pi}{2}x + \frac{x^2}{2} - 1$$

$$= -\frac{x^2}{2} + \frac{\pi}{2}x - 1 \quad \checkmark$$

$$u_t(x,0) = 0 \Rightarrow w_t(x,0) - \underbrace{\cancel{f_t(x)}}_{0} = 0 \quad \checkmark$$

$\therefore$ El cv $u(x,t) = w(x,t) + f(x)$ con $f(x) = \frac{x^2}{2} - 1$ transforma (*) en (**).

$$(**) = \begin{cases} w_{tt} = w_{xx}, \ x \in (0,\pi/2), t > 0 & (EQ') \\ w(0,t) = 0, \ w_x(\pi/2,t) = 0 & (CB') \\ w(x,0) = -\frac{x^2}{2} + \frac{\pi}{2}x - 1, \ w_t(x,0) = 0, \ x \in [0,\frac{\pi}{2}] & (CI') \end{cases}$$

b) Si $w(x,t) = M(x)N(t)$, reemplazando en (EQ'):

$$w_{tt} = w_{xx} \Rightarrow N''(t)M(x) = M''(x)N(t)$$

$$\Rightarrow \underbrace{\frac{N''(t)}{N(t)}}_{\text{Sólo depende de } t} = \underbrace{\frac{M''(x)}{M(x)}}_{\text{'' de } x} \equiv -\lambda$$

para que calce con lo que hay que llegar por enunciado

Por lo tanto

$$\frac{M''(x)}{M(x)} = -\lambda$$

$$\Rightarrow \boxed{M''(x) + \lambda M(x) = 0}$$

Ahora reemplazando en (CB'):

$$w(0,t) = 0 \Rightarrow M(0)\cancel{N(t)} = 0 \quad / \; N(t) \neq 0$$

de lo contrario sol trivial

$$\Rightarrow \boxed{M(0) = 0}$$

$$w_x\left(\tfrac{\pi}{2}, t\right) = 0 \Rightarrow M'\left(\tfrac{\pi}{2}\right)\cancel{N(t)} = 0 \quad / \; N(t) \neq 0$$

$$\Rightarrow \boxed{M'\left(\tfrac{\pi}{2}\right) = 0}$$

Resolviendo la EDO:

$$M''(x) + \lambda M(x) = 0$$

Ansatz $M(x) = e^{sx}$, reemplazando:

$$s^2 \cancel{e^{sx}} + \lambda \cancel{e^{sx}} = 0$$

$$s^2 + \lambda = 0$$

$$\Rightarrow \boxed{s = \pm\sqrt{-\lambda}}$$

$$\Rightarrow M(x) = A e^{\sqrt{-\lambda}\,x} + B e^{-\sqrt{-\lambda}\,x}$$

Comprobemos $M(0) = 0$ y $M'\left(\frac{\pi}{2}\right) = 0$

$$M(0) = 0 \Rightarrow A \cancel{e^0} + B \cancel{e^0} = 0$$

$$\Rightarrow \boxed{A = -B}$$

$$M'\left(\tfrac{\pi}{2}\right)=0 \Rightarrow A\sqrt{-\lambda}\,e^{\sqrt{-\lambda}\frac{\pi}{2}} \underbrace{+A}_{B}(+\sqrt{-\lambda})e^{-\sqrt{-\lambda}\frac{\pi}{2}}=0$$

$$\Rightarrow A\left(e^{\sqrt{-\lambda}\frac{\pi}{2}}+e^{-\sqrt{-\lambda}\frac{\pi}{2}}\right)=0$$

$$\Rightarrow \underbrace{A\cancel{=0}}_{\text{lleva a la sol trivial}} \ \vee\ \underbrace{e^{\sqrt{-\lambda}\frac{\pi}{2}}+e^{-\sqrt{-\lambda}\frac{\pi}{2}}=0}_{\text{soluciones sólo para } \lambda>0}$$

(caso $\lambda<0$ no puede satisfacer $e^{\sqrt{-\lambda}\frac{\pi}{2}}+e^{-\sqrt{-\lambda}\frac{\pi}{2}}=0$)

$\lambda>0$ | Como tenemos una raíz negativa la sol pasa a ser compuesta por exponenciales complejas

$$M(x)=Ae^{i\sqrt{\lambda}x}+Be^{-i\sqrt{\lambda}x}$$

Además antes mostramos que $A=-B$ y $e^{i\sqrt{\lambda}\frac{\pi}{2}}+e^{-i\sqrt{\lambda}\frac{\pi}{2}}=0$.

Por lo tanto:

$$\xrightarrow[A=-B]{} M(x)=A\left(e^{i\sqrt{\lambda}x}-e^{-i\sqrt{\lambda}x}\right)$$

$$=\tilde{A}\sin(\sqrt{\lambda}x)$$

$$\frac{e^{i\theta}-e^{-i\theta}}{2i}=\sin\theta \qquad \tilde{A}=\frac{A}{2i}$$

$$\Rightarrow e^{i\sqrt{\lambda}\frac{\pi}{2}}+e^{-i\sqrt{\lambda}\frac{\pi}{2}}=0$$

$$\tfrac{1}{2}\left(\right. \underbrace{\frac{e^{i\sqrt{\lambda}\frac{\pi}{2}}+e^{-i\sqrt{\lambda}\frac{\pi}{2}}}{2}}_{\cos\left(\sqrt{\lambda}\frac{\pi}{2}\right)}=0$$

$$\cos\left(\sqrt{\lambda}\tfrac{\pi}{2}\right)=0$$

$$\Rightarrow \sqrt{\lambda}\frac{\cancel{\pi}}{2}=n\cancel{\pi}+\frac{\cancel{\pi}}{2}, \quad n\geq 0$$

$$\boxed{\lambda_n=(2n+1)^2}, \quad n\geq 0$$

$$\Rightarrow M(x)=\tilde{A}\sin\left((2n+1)x\right) \qquad M(x)=\tilde{A}\sin(\sqrt{\lambda}x)$$

c) Resolviendo la otra EDO ahora que sabemos que $\lambda_n = (2n+1)^2 > 0$:

$$\frac{N''(t)}{N(t)} = -\lambda$$

$$N''(t) + \lambda N(t) = 0$$

$$\boxed{N''(t) + (2n+1)^2 N(t) = 0}$$

oscilador armonico

$$\Rightarrow N(t) = A_n \cos((2n+1)t) + B_n \sin((2n+1)t)$$

Por ppio de superposición

$$w(x,t) = \sum_{n=0}^{\infty} M(x) N(t)$$

$$\Rightarrow \underline{w(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos((2n+1)t) + B_n \sin((2n+1)t)\right] \cos((2n+1)x)}$$

d) Aplicando CI $w_t(x,0) = 0$:

$$w_t(x,0) = 0 \Rightarrow N'(0) \cancel{M(x)} = 0$$

$$\Rightarrow \cancel{-A_n(2n+1)\sin(0)} + B_n(2n+1)\cos(0) = 0$$

0, 1

$$\Rightarrow B_n(2n+1) = 0$$

$\neq 0$

$$\Rightarrow B_n = 0$$

$$\Rightarrow N(t) = A_n \cos((2n+1)t)$$

Y aplicando la otra CI:

$$w(x,0) = -\frac{x^2}{2} + \frac{\pi}{2}x - 1$$

$$\Rightarrow \sum_{n=0}^{\infty} A_n \cos(\cancel{(2n+1)\cdot 0}) \sin((2n+1)x) = -\frac{x^2}{2} + \frac{\pi}{2}x - 1$$

1

$$\underbrace{\sum_{n=0}^{\infty} \sin((2n+1)x)}_{\text{serie de cosenos de } g(x) = -\frac{x^2}{2} + \frac{\pi}{2} - 1} = -\frac{x^2}{2} + \frac{\pi}{2}x - 1$$

Multiplicando e integrando por $\left[\int_0^{\pi/2} \sin(mx)\,dx\right]$ a ambos lados de la igualdad

$2n+1=n'$

$$\int_0^{\pi/2} \sin(mx)\sum_{n=0}^{\infty} A_n \sin((2n+1)x)\,dx=\int_0^{\pi/2} g(x)\sin(mx)\,dx$$

$$\sum_{n=0}^{\infty} A_n \underbrace{\int_0^{\pi/2} \sin(mx)\sin(n'x)\,dx}_{=\begin{cases}\frac{\pi/2}{2} & m=n' \\ 0 & m\neq 0\end{cases}}=\int_0^{\pi/2} g(x)\sin(mx)\,dx$$

Como serie tiene un único término no nulo $m=n'$, entonces

$$A_{n'}\frac{\pi/2}{2}=\int_0^{\pi/2} g(x)\sin(n'x)\,dx$$

$$A_{n'}=\frac{4}{\pi}\int_0^{\pi/2} g(x)\sin(n'x)\,dx$$

$n'x=u$

$dx=\frac{du}{n'}$

$x=\frac{u}{n'}$

$x=\frac{\pi}{2}\Rightarrow u=n'\frac{\pi}{2}$

$$=\frac{4}{\pi}\left[\int_0^{\pi/2}-\frac{x^2}{2}\sin(n'x)\,dx+\int_0^{\pi/2}\frac{\pi}{2}x\sin(n'x)\,dx-\int_0^{\pi/2}\sin(n'x)\,dx\right]$$

$$=\frac{4}{\pi}\left[-\frac{1}{2n'^3}\underbrace{\int_0^{n'\frac{\pi}{2}} u^2\sin(u)\,du}_{''I_1}+\frac{\pi}{2n'^2}\underbrace{\int_0^{n'\frac{\pi}{2}} u\sin(u)\,du}_{''I_2}\underbrace{-\int_0^{\pi/2}\sin(n'x)\,dx}_{''I_3}\right]$$

$$I_1=\int_0^{\pi/2} u^2\sin(u)\,du=-u^2\cos(u)\Big|_0^{n'\pi/2}+2\cos(u)\Big|_0^{n'\pi/2}+2x\sin(u)\Big|_0^{n'\frac{\pi}{2}}$$

$$=-\left(n'\tfrac{\pi}{2}\right)^2\cancelto{0}{\cos\left(n'\tfrac{\pi}{2}\right)}+2\left(\cancelto{0}{\cos\left(n'\tfrac{\pi}{2}\right)}-\cancelto{1}{\cos(0)}\right)+2\,\frac{n'\pi}{2}\cancelto{(-1)^n}{\sin\left(n'\tfrac{\pi}{2}\right)}$$

$$=-2+n'\pi(-1)^n \quad / \; n'=2n+1$$

$$I_2=\int_0^{\pi/2} u\sin(u)\,du=\sin(u)\Big|_0^{n'\pi/2}-u\cos(u)\Big|_0^{n'\pi/2}$$

$$=\sin\left(n'\tfrac{\pi}{2}\right)-\cancelto{0}{\sin(0)}-\left(n'\tfrac{\pi}{2}\cancelto{0}{\cos\left(n'\tfrac{\pi}{2}\right)}\right)-\cancelto{0}{0\cdot\cos(0)}$$

$$=(-1)^n$$

$$I_3=-\int_0^{\pi/2}\sin(n'x)=\left[\frac{1}{n'}\cos(n'x)\right]_0^{\pi/2}=\frac{1}{n'}\cancelto{0}{\cos\left(n'\tfrac{\pi}{2}\right)}-\frac{1}{n'}\cancelto{1}{\cos(0)}$$

Reemplazando:

$$A_{n'} = \frac{4}{\pi}\left[-\frac{1}{2n'^3}\left(-2+n'\pi(-1)^n\right) + \frac{\pi}{2n'^2}(-1)^n - \frac{1}{n'} \right]$$

$$= \frac{4}{\pi}\left[\frac{1}{n'^3} \cancel{- \frac{\pi}{2n'^2}(-1)^n} \cancel{+ \frac{\pi}{2n'^2}(-1)^n} - \frac{1}{n'} \right]$$

$$= \frac{4}{\pi}\left[\frac{1}{n'^3} - \frac{1}{n'} \right]$$

$$= \frac{4}{\pi}\left[\frac{1}{(2n+1)^3} - \frac{1}{(2n+1)} \right]$$

e) La solución de (*) $u(x,t)$ por lo tanto es:

$$u(x,t) = \omega(x,t) - f(x)$$

$$= \sum_{n=0}^{\infty} \frac{4}{\pi}\left[\frac{1}{(2n+1)^3} - \frac{1}{(2n+1)}\right]\cos((2n+1)t)\sin((2n+1)x) - \frac{x^2}{2} + 1$$

P2

$f : [-\pi, \pi] \to \mathbb{R}$

$$f(x) = \begin{cases} -\frac{1}{2} & x \in [-\pi, 0] \\ \frac{1}{2} & x \in (0, \pi] \end{cases}$$

a) Calculamos la serie de f como

$$S_f = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos(nx) + b_n \sin(nx)\right)$$

donde

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \quad, n \geq 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \quad, n \geq 1$$

Primero notamos que f es impar ya que comprueba $f(x) = -f(-x)$ $\forall x \in (-\pi, \pi)$, por lo tanto $a_n = 0$ $\forall n \geq 0$.

Calculando b_n:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{impar}} \cdot \underbrace{\sin(nx)}_{\text{impar}}\,dx \qquad \text{impar} \cdot \text{impar} = \text{par}$$

$$= \frac{\cancel{2}}{\pi}\int_{0}^{\pi} \frac{1}{\cancel{2}} \sin(nx)\,dx$$

$$= \frac{1}{\pi}\left(-\frac{\cos(nx)}{n}\right)\Bigg|_0^{\pi}$$

$$= \frac{1}{\pi}\left(-\frac{\overset{(-1)^n}{\cos(n\pi)}}{n} + \frac{\overset{1}{\cos(0)}}{n}\right)$$

$$= \frac{1}{n\pi}\underbrace{\left((-1)^{n+1} + 1\right)}_{=\begin{cases} 0 & n \text{ par} \\ 2 & n \text{ impar}\end{cases}}$$

$$\Rightarrow \boxed{S_f = \sum_{n \text{ impar}}^{\infty} \frac{2}{n\pi}\sin(nx)}$$

b) Recordamos Parseval

$$\int_{-\pi}^{\pi} (f(x))^2 dx = \pi\left(\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right)$$

Aplicado para f y los bn calculados antes:

$$\sum_{n=1}^{\infty} b_n^2 = \int_{-\pi}^{\pi} (f(x))^2 dx \quad \text{par}$$

$$\sum_{n \text{ impar}}^{\infty}\left(\frac{2}{n\pi}\right)^2 = 2\int_0^{\pi}\frac{1}{4}\,dx$$

$$\cdot\frac{\pi^2}{4} \qquad n = 2n'-1$$

$$\boxed{\sum_{n'=1}^{\infty}\frac{1}{(2n'-1)^2} = \frac{\pi^3}{8}}$$

c) Como f continua por trozos:

$$S_f = \frac{1}{2}\left(f(0)^- + f(0)^+\right)$$

$$= \frac{1}{2}\left(-\frac{1}{2} + \frac{1}{2}\right)$$

$$= 0$$

Para d) y e) ver pauta P1C3 Otoño 2023. Estos ítem no evalúan exactamente los contenidos del curso, se resuelven más bien con pillería. Nota: la d) ocupa una sumatoria telescópica (lo que no se dice en la pauta).