P_1

a) $f(x) = e^{-a|x|}$, $a > 0$.

Por definición de TF:

$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} e^{-isx}\, dx \quad |x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} e^{-a(-x)} e^{-isx}\, dx + \int_{0}^{\infty} e^{-ax} e^{-isx}\, dx \right)$$

CV $x' = -x$

$dx' = -dx$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{\infty}^{0} e^{-ax'} e^{isx'} (-dx') \right) + \int_{0}^{\infty} e^{-ax} e^{-isx}\, dx \Big)$$

cancelamos - dando vuelta límites

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{0}^{\infty} e^{-ax'} e^{isx'}\, dx + \int_{0}^{\infty} e^{-ax} e^{-isx}\, dx \right.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ax} \underbrace{\left(e^{isx} + e^{-isx} \right)}_{= 2\cos(sx)} dx$$

$$= \frac{2}{\sqrt{2\pi}} \underbrace{\int_{0}^{\infty} e^{-ax} \cos(sx)\, dx}_{\text{IPP}} \quad I$$

IPP

$u = e^{-ax} \xrightarrow{\frac{d}{dx}} du = -a e^{-ax}$

$dv = \cos(sx)\, dx \xrightarrow{\int} v = \frac{\sin(sx)}{s}$

$$= \sqrt{\frac{2}{\pi}} \left[e^{-ax} \frac{\sin(sx)}{s} \Big|_0^{\infty} + \int_0^{\infty} \frac{\sin(sx)}{s} + a e^{-ax}\, dx \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[\lim_{x \to \infty} \cancel{\left(\frac{e^{-ax} \sin(sx)}{s} \right)}^{\to 0} - \cancel{\frac{e^{0} \sin(0)}{s}}^{\to 0} + \frac{a}{s} \int_0^{\infty} e^{-ax} \sin(sx)\, dx \right]$$

IPP de nuevo

$u = e^{-ax} \xrightarrow{()'} du = -a e^{-ax}$

$dv = \sin(sx)\, dx \xrightarrow{\int} v = -\frac{\cos(sx)}{s}$

$$= \frac{2}{\sqrt{2\pi}}\left[\underbrace{\frac{a}{s}\left(-e^{-ax}\right)\frac{\cos(sx)}{s}\Big|_0^\infty - \frac{a^2}{s}\int_0^\infty e^{-ax}\cos(sx)\,dx}_{=I}\right]$$

$$\int_0^\infty e^{-ax}\cos(sx)\,dx = I$$

$$\Rightarrow I = \frac{a}{s^2}\left(-\lim_{x\to\infty} \cancel{e^{-ax}\frac{\cos(sx)}{s}}_{\to 0} + e^0 \cancel{\cos(0)}_{1}\right) - \frac{a^2}{s}I$$

$$I = \frac{a}{s^2} - \frac{a^2}{s^2}I$$

$$\Rightarrow I = \frac{a}{s^2}\left(1+\frac{a^2}{s^2}\right)^{-1}$$

$$= \frac{a}{\cancel{s^2}}\left(\frac{\cancel{s^2}}{s^2+a^2}\right)$$

$$= \frac{a}{s^2+a^2}$$

$$\Rightarrow \hat{f}(s) = \sqrt{\frac{2}{\pi}}\,\frac{a}{s^2+a^2}$$

b) $g(x) = e^{-ax^2}, a>0$

Por definición:

$$\hat{g}(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2}e^{-isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(ax^2+isx)}\,dx$$ completamos cuadrado

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(\underbrace{ax^2+isx+\left(\frac{is}{2\sqrt{a}}\right)^2}_{=\left(\sqrt{a}x+\frac{is}{2\sqrt{a}}\right)^2}-\left(\frac{is}{2\sqrt{a}}\right)^2\right)}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(\sqrt{a}x+\frac{is}{2\sqrt{a}}\right)^2} e^{\left(\frac{is}{2\sqrt{a}}\right)^2}\,dx$$

no depende de x, se puede sacar

$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{4a}}\underbrace{\int_{-\infty}^{\infty} e^{-\left(\sqrt{a}x+\frac{is}{2\sqrt{a}}\right)^2}\,dx}_{=I}$$

$$I = \int_{-\infty}^{\infty} e^{-\left(\sqrt{a}x+\frac{is}{2\sqrt{a}}\right)^2}\,dx \quad \left\{\begin{array}{l} u = \sqrt{a}x + \frac{is}{2\sqrt{a}} \\ du = \sqrt{a}\,dx \end{array}\right.$$

$$= \frac{1}{\sqrt{a}}\underbrace{\int_{-\infty}^{\infty} e^{-u^2}\,du}_{=\sqrt{\pi}}$$ integral gaussiana

$= \sqrt{\frac{\pi}{a}}$

$$\Rightarrow \hat{g}(s) = \frac{1}{\sqrt{2\cancel{\pi}}}\, e^{-s^2/4a}\sqrt{\frac{\cancel{\pi}}{a}}$$

$$= \boxed{\frac{1}{\sqrt{2a}}\, e^{-s^2/4a}}$$

c) $h(x)=\frac{a}{a^2+x^2}$

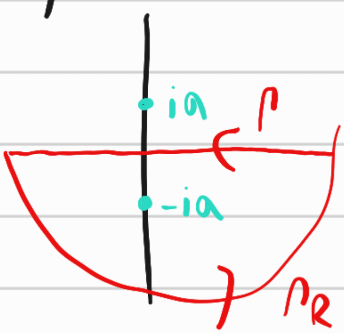
2 formas de hacerla:

1) Por definición y residuos:

$$\hat{h}(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{a}{a^2+x^2}e^{-isx}\,dx$$

Sea $f(z)=\frac{ae^{-isz}}{z^2+a^2}$. Polos en $z_1 = ia \wedge z_2 = -ia$.
Para formar el contorno de integración analizamos el término e^{-isz}:

Si $s>0$, $z=Re^{i\theta}=R\cos\theta+iR\sin\theta \Rightarrow e^{-isz} = \underbrace{e^{-isR\cos\theta}}_{\text{acotado cuando } R\to\infty}\ \underbrace{e^{sR\sin\theta}}_{\text{puede diverger o irse a cero cuando } R\to\infty}$



$C=\Gamma\cup\Gamma_R$

$$\int_C f(z)\,dz = \underbrace{\int_\Gamma f(z)\,dz}_{\substack{z=x \\ R\to\infty}\ \to\ \int_{-\infty}^{\infty}\frac{ae^{-isx}}{a^2+x^2}dx} + \underbrace{\int_{\Gamma_R} f(z)\,dz}_{\to 0 \text{ cuando } R\to\infty}$$

ya que $f(z)\propto e^{sR\sin\theta}$, $\sin\theta<0$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{ae^{-isx}}{a^2+x^2}\,dx = -\cancel{2}\pi\cancel{i}\,\underbrace{\text{Res}(f,-ia)}$$

$$\lim_{z\to -ia}\cancel{(z+ia)}\frac{ae^{-isz}}{\cancel{(z+ia)}(z-ia)}$$

$$= \frac{\cancel{a}\, e^{-is(-ia)}}{\cancel{-2i}\cancel{a}}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{ae^{-isx}}{a^2+x^2}\,dx = \pi e^{-sa}$$

$$\Rightarrow \hat{h}(s) = \frac{1}{\sqrt{2\pi}}\pi e^{-sa} = \sqrt{\frac{\pi}{2}}\, e^{sa},\quad s>0$$

Análogamente para $s<0$, cerrando C por arriba (se encierra ia en vez $-ia$, donde $\sin\theta > 0$):

$$\hat{h}(s) = \sqrt{\frac{\pi}{2}}\, e^{sa} \quad , \quad s<0$$

Lo que se puede escribir en un solo termino como:

$$\boxed{\hat{h}(s) = \sqrt{\frac{\pi}{2}}\, e^{-a|s|}} \quad , \quad \forall s \in \mathbb{R}$$

2) Ocupando el resultado del item a):

$$\hat{h}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{a}{a^2+x^2} e^{-isx}\, dx \qquad \begin{matrix} u=-x \\ du=-dx \end{matrix}$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{a}{a^2+u^2} e^{isu}\, du}_{\mathcal{F}^{-1}\left(\frac{a}{a^2+u^2}\right)(s)}$$

$$\Rightarrow \hat{h}(s) = \mathcal{F}^{-1}\left(\frac{a}{a^2+(\cdot)^2}\right)(s)$$

Ocupando a):

$$\mathcal{F}\left[e^{-a|x|}\right](s) = \sqrt{\frac{2}{\pi}}\, \frac{a}{a^2+s^2}$$

$\mathcal{F}^{-1}[\cdot](x)$

$$\cancel{\mathcal{F}^{-1}}\left[\cancel{\mathcal{F}}\left[e^{-a|x|}\right](s)\right](x) = \sqrt{\frac{2}{\pi}}\, \mathcal{F}^{-1}\left[\frac{a}{a^2+s^2}\right](x)$$

$$e^{-a|x|} = \sqrt{\frac{2}{\pi}}\, \mathcal{F}^{-1}\left[\frac{a}{a^2+s^2}\right](x)$$

$$\Rightarrow \mathcal{F}^{-1}\left[\frac{a}{a^2+s^2}\right](x) = \sqrt{\frac{\pi}{2}}\, e^{-a|x|}$$

si intercambiamos s por x (variables mudas)

$$\Rightarrow \hat{h}(s) = \mathcal{F}^{-1}\left[\frac{a}{a^2+x^2}\right](s) = \boxed{\sqrt{\frac{\pi}{2}}\, e^{-a|s|}}$$

d) $p(x) = \mathbb{1}_{[-b,b]} = \begin{cases} 1 & \text{si } x \in [-b,b] \\ 0 & \text{si } |x| = b \end{cases}$

$$\hat{p}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbb{1}_{[-b,b]}(x) e^{-isx} dx$$

aplicando def de $\mathbb{1}_{[-b,b]}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-b}^{b} e^{-isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[-\frac{1}{is} e^{-isx} \right]_{-b}^{b}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{s} \underbrace{\left[\frac{e^{isb} - e^{-isb}}{i} \right] \frac{2}{2}}_{\sin(sb)}$$

$$\hat{p}(s) = \sqrt{\frac{2}{\pi}} \frac{\sin(sb)}{s}$$

P2

a) Pdq

$$\underbrace{\int_{-\infty}^{\infty} f(x) g^*(x) dx}_{LHS} = \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(s) \hat{g}^*(s) ds}_{RHS}$$

Partiendo de LHS:

$$\int_{-\infty}^{\infty} f(x) g^*(x) dx = \int_{-\infty}^{\infty} f(x) \left[\left(\hat{g}(s) \right)^{\vee} \right]^* (x) ds$$

$$= \int_{-\infty}^{\infty} f(x) \frac{1}{2\pi} \underbrace{\left(\int_{-\infty}^{\infty} \hat{g}(s) e^{isx} ds \right)^*}_{\text{conjugación}} dx$$

Fubini

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left(\int_{-\infty}^{\infty} \hat{g}^*(s) e^{-isx} ds \right) dx$$

$$= \int_{-\infty}^{\infty} \hat{g}^*(s) \underbrace{\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-isx} dx \right)}_{= \hat{f}(s)} ds$$

$$= \int_{-\infty}^{\infty} \hat{f}(s) \hat{g}^*(s) ds$$

b) Ahora para obtener la identidad de Parseval basta tomar g= y aplicar Plancherel:

$$\int_{-\infty}^{\infty} f(x) f^*(x) dx = \int_{-\infty}^{\infty} \hat{f}(s) \hat{f}^*(s) ds$$

$$\boxed{\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(s)|^2 ds}$$

$zz^* = |z|^2$

P3

Queremos resolver la ec diferencial de Airy dada por:

$$y''(x) - x y(x) = 0$$

Aplicamos TF a ver qué pasa:

$$\mathcal{F}[y''](k) - \mathcal{F}[xy](k) = 0$$

$$(ik)^2 \mathcal{F}[y](k) - \mathcal{F}[xy(x)](k) = 0$$

analicemos esto con más detalle

$$\mathcal{F}[xy(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x y(x) e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y(x) \underbrace{[x e^{-ikx}]}_{= i \frac{d}{dk} e^{ikx}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y(x)\, i \frac{d}{dk} (e^{-ikx}) dx$$

$$= i \frac{d}{dk} \Big(\underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y(x) e^{-ikx} dx}_{= \hat{y}(k)} \Big)$$

sale fuera de la integral pq lo otro no depende de k

$$= i \frac{d}{dk} \hat{y}(k)$$

Por lo tanto la ec de Airy al aplicar TF toma la sgte forma

$$-k^2 \hat{y}(k) - i \hat{y}'(k) = 0$$

$$\hat{y}'(k) = i k^2 \hat{y}(k)$$

$$\Rightarrow \hat{y}(k) = A e^{i \int k^2 dk}$$

$$\boxed{= A e^{i \frac{k^3}{3}}}$$

Aplicando anti TF tenemos la sol general:

$$y(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A e^{ik^3/3} e^{ikx}\, dk$$

$$= \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\left(kx + \frac{k^3}{3}\right)}\, dk$$

$$= \underbrace{\frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos\left(kx + \frac{k^3}{3}\right) dk}_{\text{si } A = C\frac{1}{\sqrt{2\pi}}} + \underbrace{\frac{iA}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin\left(kx + \frac{k^3}{3}\right) dk}_{D\, Bi(x)}$$

$$= \underbrace{C \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos\left(kx + \frac{k^3}{3}\right) dk}_{= C\, A_i(x)}$$

$$\Rightarrow y(x) = c_1 A_i(x) + c_2 Bi(x)$$ / solución más general

solución particular