$\boxed{P_1}$

$$f(x) = x^2 \quad , x \in [-\pi, \pi]$$

Encontrar la serie de Fourier de $f(x)$ implica encontrar los coefs a_n y b_n que hacen que

$$S_f = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + \cancel{b_n} \sin(nx) \right)$$

$\to 0$ dado que $f(x)$ par

Donde los coefs cumplen:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,dx \qquad n \in \{0, 1, 2, \ldots\}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx)\,dx$$

$n = 0$

$$\Rightarrow \frac{a_0}{2} = \frac{\cancel{2}}{\cancel{2}\pi} \int_0^{\pi} x^2\,dx$$

$$= \frac{1}{\pi} \left. \frac{x^3}{3} \right|_0^{\pi}$$

$$= \boxed{\frac{\pi^2}{3}}$$

$n \neq 0$

$$a_n = \frac{1}{\pi} \underbrace{\int_0^{\pi} x^2 \cos(nx)\,dx}_{\text{IPP (int. por partes)}}$$

$$\boxed{\begin{array}{l} u = x^2 \Rightarrow du = 2x\,dx \\ dv = \cos(nx)\,dx \Rightarrow v = \frac{1}{n} \sin(nx) \end{array}}$$

$$a_n = \frac{2}{\pi} \left(\cancel{\left. \frac{x^2}{n} \sin(nx) \right|_0^{\pi}}^{\,0} - \frac{2}{n} \underbrace{\int_0^{\pi} x \sin(nx)}_{\text{IPP de nuevo}} \right)$$

$$\boxed{\begin{array}{l} u = x \Rightarrow du = dx \\ dv = \sin(nx) \Rightarrow v = -\frac{1}{n} \cos(nx) \end{array}}$$

$$\Rightarrow a_n = \frac{1}{\pi} \left(\left. \frac{2x}{n^2} \cos(nx) \right|_0^{\pi} + \frac{2}{n} \cdot \left(\frac{-1}{n} \right) \int_0^{\pi} \cos(nx) \right)$$

$$= \frac{1}{\pi} \left(\frac{2\pi}{n^2} \underbrace{\cos(n\pi)}_{(-1)^n} - \frac{2}{n^2} \frac{1}{n} \cancel{\left. \sin(n\pi) \right|_0^{\pi}}^{\,0} \right)$$

$$= \frac{1}{\cancel{\pi}} \cdot \frac{2\cancel{\pi}}{n^2}(-1)^n = \boxed{\frac{2}{n^2}(-1)^n}$$

Por lo que la serie es:

$$\boxed{S_f = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos(nx)}$$

Y dado que x^2 es continua y L^2 se tiene que $S_f = f(x)$

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos(nx) \quad *$$

b) Evaluando para $x = \pi$ $*$:

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}\underbrace{(-1)^n \overbrace{\cos(n\pi)}^{(-1)^n}}_{1}$$

$$\Rightarrow 4\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2 - \frac{\pi^2}{3}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{\cancel{4}\,2} \frac{\cancel{2}\pi^2}{3} = \frac{\pi^2}{6}}$$

P2

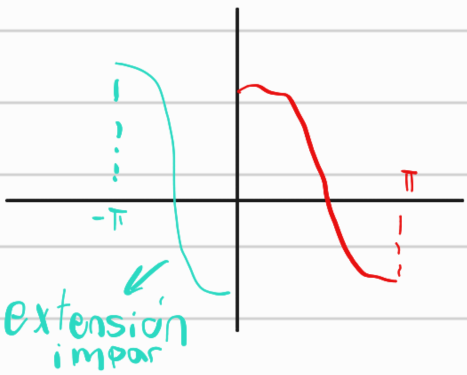
$$f(x) = \cos(x) \qquad x \in [0, \pi]$$

$$\text{Pdq} \quad f(x) = \sum_{n=1}^{\infty} \frac{8}{\pi} \frac{n}{4n^2 - 1} \sin(2nx)$$

Primero notamos que la expansión que se realiza es una serie en senos, las cuales se dan para expansiones de funciones impares. Por lo tanto extendemos $f(x)$ tq sea impar en $x \in [-\pi, \pi]$

$$\hat{f}(x) = \begin{cases} f(x) & x \in [0, \pi] \\ -f(x) & x \in [-\pi, 0) \end{cases}$$

$$\Rightarrow S_{\hat{f}} = \sum_{n=1}^{\infty} b_n \sin(nx)$$



donde los coefs b_n se calculan como

$$b_n = \frac{1}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$\sin(nx-x) = \cos(x)\sin(nx) - \cos(x)\sin(nx)$$
$$(+)\ \sin(nx+x) = \cos(x)\sin(nx) + \cos(x)\sin(nx)$$
$$\Rightarrow \sin(nx-x)+\sin(nx+x) = 2\cos(x)\sin(nx)$$

$$= \frac{2}{\pi}\int_0^{\pi}\cos(x)\sin(nx)\,dx$$

$$= \frac{\cancel{2}}{\pi}\frac{1}{\cancel{2}}\int_0^{\pi}\left[\sin(nx-x)+\sin(nx+x)\right]dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi}\sin(x(n-1))\,dx + \int_0^{\pi}\sin(x(n+1))\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n-1}\left(\cancel{\cos(0)}^{1} \underbrace{- \cos(\pi(n-1))}_{(-1)\,(-1)^{n-1} = (-1)^n}\right) + \frac{1}{n+1}\left(\cancel{\cos(0)}^{1} \underbrace{- \cos(\pi(n+1))}_{(-1)(-1)^{n+1} = (-1)^{n+2} = (-1)^n(-1)^2 = (-1)^n}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{1+(-1)^n}{n-1} + \frac{1+(-1)^n}{n+1}\right]$$

$$= \frac{1+(-1)^n}{\pi}\left[\underbrace{\frac{1}{n-1}+\frac{1}{n+1}}_{\frac{2n}{n^2-1}}\right]$$

$$1+(-1)^n = \begin{cases} 2 & n \text{ par} \\ 0 & n \text{ impar}\end{cases}$$

Sea $n' = 2n$, n par

$$\Rightarrow b_{2n=n'} = \frac{2\cdot 2(2n)}{\pi}\frac{1}{4n^2-1}$$

$$= \boxed{\frac{8}{\pi}\frac{n}{4n^2-1}}$$

$f(x) \in L^2$
(cuadrado integrable)

$$f(x) = S_f^{\sim} = \sum_{n=1}^{\infty}\frac{8}{\pi}\frac{n}{4n^2-1}\sin(2nx)\ (*)$$

Ahora bien, para demostrar el valor de la serie

$$\frac{1}{2^2-1} - \frac{3}{6^2-1} + \frac{5}{10^2-1} - \frac{7}{14^2-1} + \cdots$$

Evaluamos $(*)$ en un pto adecuado. Si $x = \frac{\pi}{4}$

$$\frac{\sqrt{2}}{2} = \cos\left(\frac{\pi}{4}\right) = \sum_{n=1}^{\infty}\frac{8}{\pi}\frac{n}{4n^2-1}\sin\left(\cancel{2}n\frac{\pi}{\cancel{4}_2}\right)$$

$$\frac{\sqrt{2}}{2} = \sum_{n=1}^{\infty} \frac{8}{\pi} \frac{n}{4n^2-1} \sin\left(n\frac{\pi}{2}\right)$$

$$\sin\left(n\frac{\pi}{2}\right) = \begin{cases} 0 & n \text{ par} \\ (-1)^{k+1} & n = 2k-1 \text{ impar} \end{cases}$$

$\cdot \frac{\pi}{8}$ $n=2k-1$

$$\frac{\pi\sqrt{2}}{16} = \sum_{k=1}^{\infty} \frac{2k-1}{4(2k-1)^2-1}(-1)^{k+1}$$

$$4(2k-1)^2-1 = (2\cdot(2k-1))^2-1$$

$$= \frac{1}{(2\cdot 1)^2-1} - \frac{3}{(2\cdot 3)^2-1} + \frac{5}{(2\cdot 5)^2-1} - \frac{7}{(2\cdot 7)^2-1} + \dots \quad \blacksquare$$

P3

Sea $f: [-\pi,\pi] \to \mathbb{R}$ clase C^1 definida por:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right)$$

a) Pdq) Identidad de Parseval:

$$\int_{-\pi}^{\pi}(f(x))^2dx = \pi\frac{a_0^2}{2} + \pi\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)$$

Partiendo del lado izq:

$$\int_{-\pi}^{\pi}(f(x))^2dx = \int_{-\pi}^{\pi}\left[f(x)\cdot \underbrace{\left(\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx)+b_n\sin(nx)\right)\right)}_{=f(x)}\right]dx$$

$f(\pi)=f(-\pi)$ y f continua ($\in C^1$) $\Rightarrow$ converge en media cuadrática y se puede intercambiar Σ por $\int$

$$= \frac{a_0}{2}\underbrace{\int_{-\pi}^{\pi}f(x)\,dx}_{=\pi a_0} + \int_{-\pi}^{\pi}f(x)\sum_{n=1}^{\infty}\left(a_n\cos(nx)+b_n(\sin(nx))\right)dx$$

$$= \pi\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(\int_{-\infty}^{\infty}f(x)\left(a_n\cos(nx)+b_n\sin(nx)\right)dx\right.$$

$$= \pi\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n\underbrace{\int_{-\infty}^{\infty}f(x)\cos(nx)dx}_{=\pi a_n} + b_n\underbrace{\int_{-\infty}^{\infty}f(x)\sin(nx)dx}_{=\pi b_n}\right)$$

$$= \pi \frac{a_0^2}{2} + \pi \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2 \right) \qquad \blacksquare$$

b) Pdq $\displaystyle \int_{-\pi}^{\pi} \left(f'(x)\right)^2 dx = \pi \sum_{n=1}^{\infty} n^2 \left(a_n^2 + b_n^2 \right)$

Si $g(x) = f'(x)$:

$$g(x) = \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right) \right)'$$

$$= \sum_{n=1}^{\infty} \left(a_n \left(-\sin(nx) n \right) + b_n n \cos(nx) \right)$$

$$= \sum_{n=1}^{\infty} \left(\underbrace{n b_n}_{= a_n'} \cos(nx) + \underbrace{(-n a_n)}_{= b_n'} \sin(nx) \right)$$

Como $f'(\pi) = f'(-\pi)$ y continua (clase C^1), entonces se puede aplicar Parseval para $g(x)$ con $a_0' = 0$, $a_n' = n b_n$ y $b_n' = -n a_n$:

$$\int_{-\pi}^{\pi} \left(f'(x) \right) dx = \int_{-\pi}^{\pi} \left(g(x) \right)^2 dx$$

Parseval para $g(x)$

$$= \pi \underbrace{\cancel{\frac{a_0'^2}{2}}}_{a_0' = 0} + \pi \sum_{n=1}^{\infty} \left(\underbrace{a_n'^2}_{a_n' = n b_n} + \underbrace{b_n'^2}_{b_n' = -n a_n} \right)$$

$$= \pi \sum_{n=1}^{\infty} n^2 \left(a_n^2 + b_n^2 \right) \qquad \blacksquare$$