Universidad de Chile
Facultad de Ciencias Físicas y Matemáticas
Departamento de Ingeniería Matemática
MA2001-2 – Cálculo en Varias Variables

Auxiliar Extra: Preparación Control 2

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P1 [C2 2016-1 - Daniilidis]
Considere $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ y $g:\mathbb{R}\longrightarrow\mathbb{R}^2$ dos funciones definidas por:

$$f(x,y)=\begin{cases}\frac{x^3y}{x^4+y^2} & \text{si } (x,y)\neq(0,0)\\ 0 & \text{si } (x,y)=(0,0)\end{cases}\qquad g(t)=\begin{cases}\left(t,t^2\operatorname{sen}\left(\frac{1}{t}\right)\right) & \text{si } t\neq 0\\ (0,0) & \text{si } t=0\end{cases}$$

(a) Determine para qué direcciones existe la derivada direccional de f en $(x,y)=(0,0)$.

(b) Encuentre las derivadas parciales de f donde existan.

(c) Muestre que g es diferenciable en $t=0$.

(d) Estudie la diferenciabilidad de $(f\circ g)$ en $t=0$. Concluya acerca de la diferenciabilidad de f en $(x,y)=(0,0)$.

P2 [C2 2010-1 - Correa]
Considere las funciones $f:\mathbb{R}^3\to\mathbb{R}^2, g:\mathbb{R}^2\to\mathbb{R}, h:U\subset\mathbb{R}^2\to\mathbb{R}$ definidas por

$$f(x,y,z)=(x\sin(z),y\cos(z)),\quad g(x,y)=y^2\cos(x)$$

$$h(x,y)=-y\ln(\cos(x))$$

A partir de estas funciones, se define $\varphi:V\subset\mathbb{R}^2\to\mathbb{R}^2$ por

$$\varphi(x,y)=f(x+y,g(x,y),h(y,x)).$$

Calcule $D\varphi(x,y)$ y en particular concluya que

$$D\varphi(0,1)=\begin{bmatrix}-\ln(\cos(1)) & 0\\ 0 & 2\end{bmatrix}$$

P3 [C2 2015-2 - Soto]
Sea $u:\mathbb{R}^2\backslash\{(0,0)\}\to\mathbb{R}$ una función de clase $\mathcal{C}^2$ y sea $v:\mathbb{R}^2\backslash\{(0,0)\}\to\mathbb{R}$ definida por

$$v(x,y)=u\left(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2}\right).$$

Decimos que u es armónica si satisface la *ecuación de Laplace* en su dominio, en este caso

$$\Delta u=0 \text{ en } \mathbb{R}^2\backslash(0,0)$$

donde $\Delta u=\frac{\partial u^2}{\partial x^2}+\frac{\partial u^2}{\partial y^2}$. Demuestre que u es armónica en $\mathbb{R}^2\backslash\{(0,0)\}$ sí y solamente sí v es armónica en $\mathbb{R}^2\backslash\{(0,0)\}$.

$\boxed{\textbf{P4}}$ **[Ecuación de Ondas] [C2 2016-1 - Del Pino]**
Suponga que $\varphi : \mathbb{R}^2 \to \mathbb{R}, \varphi = \varphi(x,t)$ es una función de clase C^2 en $\mathbb{R}^2$ que para $c > 0$ satisface

$$\frac{\partial^2 \varphi}{\partial t^2} = c^2 \frac{\partial^2 \varphi}{\partial x^2} \quad \forall (x,t) \in \mathbb{R}^2 \tag{1}$$

Definimos la función auxiliar $\psi(u,v) = \varphi\left(\frac{u+v}{2}, \frac{u-v}{2c}\right)$, buscamos resolver (1), para ello se proponen los siguientes pasos

(a) Demuestre que

$$\frac{\partial^2 \psi}{\partial u \partial v}(u,v) = 0 \quad \forall (u,v) \in \mathbb{R}^2.$$

(b) Deduzca que existen $f, g : \mathbb{R} \to \mathbb{R}$ de clase C^2 tales que

$$\varphi(x,t) = f(x+ct) + g(x-ct).$$

(c) Suponiendo además que $\varphi(x,0) = \varphi_0(x)$ y que $\frac{\partial \varphi}{\partial t}(x,0) = \varphi_1(x)$, muestre que

$$\varphi(x,t) = \frac{1}{2}\left[\varphi_0(x+ct) + \varphi_0(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} \varphi_1(s)ds\right]$$

$\boxed{\textbf{P5}}$ **[Propuesto] [C2 2011-1 - Coordinado]**
Sea $f : \mathbb{R}^2 \to \mathbb{R}$ una función de clase $\mathcal{C}^2$ una solución de la *ecuación de Laplace*

$$\frac{\partial^2 f}{\partial u^2}(u,v) + \frac{\partial^2 f}{\partial v^2}(u,v) = 0.$$

Sean $u(x,y), v(x,y)$ funciones diferenciables que satisfacen las *ecuaciones de Cauchy-Riemann*

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y), \quad \frac{\partial v}{\partial x}(x,y) = -\frac{\partial u}{\partial y}(x,y) \tag{2}$$

Demuestre que la función definida por

$$g(x,y) = f(u(x,y), v(x,y))$$

también satisface la *ecuación de Laplace*:

$$\frac{\partial^2 g}{\partial x^2}(x,y) + \frac{\partial^2 g}{\partial y^2}(x,y) = 0.$$

$\boxed{\textbf{P6}}$ **[Propuesto] [C2 2017-1 - Daniilidis]** Sea $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ una función diferenciable. Se definen las funciones $G : \mathbb{R} \to \mathbb{R}$ $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ por:

$$G(t) = f(t,t,\ldots)$$
$$h(x_1,\ldots,x_n) = G\left(\frac{x_1 + \ldots + x_n}{n}\right)$$

(a) Calcule $G'(t)$ y $\nabla h(x_1,\ldots,x_n)$ en terminos de las derivadas parciales de f.

(b) Demuestre que

$$\forall x \in \mathbb{R}, \quad \sum_{i=1}^{n} \frac{\partial (f-h)}{\partial x_i}(x,\ldots,x) = 0$$

Pauta Aux Extra

P1 **[C2 2016-1 - Daniilidis]**
Considere $f:\mathbb{R}^2 \longrightarrow \mathbb{R}$ y $g:\mathbb{R} \longrightarrow \mathbb{R}^2$ dos funciones definidas por:

$$f(x,y) = \begin{cases} \frac{x^3 y}{x^4+y^2} & \text{si } (x,y) \neq (0,0) \\ 0 & \text{si } (x,y) = (0,0) \end{cases} \qquad g(t) = \begin{cases} \left(t, t^2 \operatorname{sen}\left(\frac{1}{t}\right)\right) & \text{si } t \neq 0 \\ (0,0) & \text{si } t = 0 \end{cases}$$

(a) Determine para qué direcciones existe la derivada direccional de f en $(x,y) = (0,0)$.

(b) Encuentre las derivadas parciales de f donde existan.

(c) Muestre que g es diferenciable en $t=0$.

(d) Estudie la diferenciabilidad de $(f \circ g)$ en $t=0$. Concluya acerca de la diferenciabilidad de f en $(x,y) = (0,0)$.

(a) Estudiamos la derivada de f en dirección $d = (d_1, d_2)$ en $(0,0)$

$$\mathcal{D}f((0,0);d) = \lim_{t\to 0} \frac{f((0,0)+td) - f((0,0))}{t}$$

$$= \lim_{t\to 0} \frac{f(td_1, td_2)}{t} = \lim_{t\to 0} \frac{(td_1)^3(td_2)}{(td_1)^4+(td_2)^2}$$

$$= \lim_{t\to 0} \frac{1}{t} \cdot \frac{t^4 d_1^3 d_2}{t^2(t^2 d_1^4 + d_2^2)} = \lim_{t\to 0} \frac{t\, d_1^3 d_2}{t^2 (d_1)^2 + (d_2)^2}$$

Notamos que si $d_2 = 0 \Rightarrow \mathcal{D}f((0,0),d) = 0$ (el numerador se anula)

Si $d_2 \neq 0$, vemos que

$$\lim_{t\to 0} \left| \frac{t\, d_1^3 d_2}{t^2(d_1)^2+(d_2)^2} \right| = \lim_{t\to 0} \left| \underbrace{t}_{\to 0} \underbrace{\frac{d_1^3 d_2}{t^2(d_1)^2+(d_2)^2}}_{\to \text{ converge}} \right| = 0$$

Donde usamos que $\frac{d_1^3 d_2}{t^2 d_1^2 + d_2^2} \longrightarrow \frac{d_1^3 d_2}{d_2^2} \in \mathbb{R}$ $(d_2 \neq 0)$

Entonces, en cualquier caso,

$$\mathcal{D}f((0,0),d) = 0 \quad \forall d \in \mathbb{R}^2$$

(6) Estudiamos por separado los casos $(x,y) \neq 0$ y $(x,y) = (0,0)$

• Caso 1: $(x,y) \neq 0$

•) $$\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}\left[\frac{x^3 y}{x^4+y^2}\right] = \frac{3x^2y(x^4+y^2) - x^3y(4x^3)}{(x^4+y^2)^2}$$

$$= \frac{3x^2y^3 - x^6y}{(x^4+y^2)^2}$$

•) $$\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y}\left[\frac{x^3 y}{x^4+y^2}\right] = \frac{x^3\cdot(x^4+y^2) - x^3y\cdot 2y}{(x^4+y^2)^2}$$

$$= \frac{x^7 - x^3y^2}{(x^4+y^2)^2}$$

• Caso 2: $(x,y) = (0,0)$

•) $$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f((0,0)+(t,0)) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{t^3\cdot 0}{t^4+0^2}\cdot\frac{1}{t} = 0$$

•) $$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f((0,0)+(0,t)) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{1}{t}\cdot\frac{0\cdot t}{0^4+t^2} = 0$$

Por lo que $\frac{\partial f}{\partial x}$ y $\frac{\partial f}{\partial y}$ existen en todo $\mathbb{R}^2$

c) Calculamos la derivada de g en $t=0$

$$g'(0) = \lim_{t\to 0} \frac{g(0+t)-g(0)}{t} = \lim_{t\to 0} \frac{1}{t}\cdot\left(t\,,\,t^2\sin\left(\frac{1}{t}\right)\right)$$

$$= \lim_{t\to 0} \left(1\,,t\sin\left(\frac{1}{t}\right)\right) = (1\,,\,0)$$

Luego g es diff en $t=0$ con $g'(0)=(1,0)$

d) Importante: No sabemos si f es diff en $(0,0)$, luego no podemos usar regla de la cadena para $(f\circ g)$ en $t=0$

Veamos la forma que tiene $(f\circ g)(t)$.

Si $t\neq 0$, $(f\circ g)(t) = f\left(t,t^2\sin\left(\frac{1}{t}\right)\right)$

$$= \frac{t^3\cdot t^2\sin\left(\frac{1}{t}\right)}{t^4+t^4\sin^2\left(\frac{1}{t}\right)} = \frac{t\sin\left(\frac{1}{t}\right)}{1+\sin^2\left(\frac{1}{t}\right)}$$

Si $t=0$, $(f\circ g)(0) = f(0,0)=0$

Veamos $(f\circ g)'(0)$ (si existe)

$$= \lim_{t\to 0} \frac{(f\circ g)(t)-(f\circ g)(0)}{t}$$

$$= \lim_{t\to 0} \frac{\cancel{t}\sin\left(\frac{1}{t}\right)}{1+\sin^2\left(\frac{1}{t}\right)}\cdot\frac{1}{\cancel{t}} = \lim_{t\to 0}\frac{\sin\left(\frac{1}{t}\right)}{1+\sin^2\left(\frac{1}{t}\right)}$$

Notemos que $\lim\limits_{t\to 0} \dfrac{\sin\left(\frac{1}{t}\right)}{1+\sin^2\left(\frac{1}{t}\right)}$ no existe, para ello

Considerando $(t_n) = \dfrac{1}{2\pi n} \xrightarrow[n\to+\infty]{} 0$

$$\frac{\overbrace{\sin(2\pi n)}^{0}}{1+\underbrace{\sin^2(2\pi n)}_{0}} = 0 \xrightarrow[n\to+\infty]{} 0$$

Considerando $(t_n) = \dfrac{1}{2\pi n + \frac{\pi}{2}} \xrightarrow[n\to+\infty]{} 0$

$$\frac{\overbrace{\sin\left(2\pi n + \frac{\pi}{2}\right)}^{1}}{1+\underbrace{\sin^2\left(2\pi n+\frac{\pi}{2}\right)}_{1}} = \frac{1}{2} \xrightarrow[n\to+\infty]{} \frac{1}{2}$$

Luego $\lim\limits_{t\to 0} \dfrac{\sin\left(\frac{1}{t}\right)}{1+\sin^2\left(\frac{1}{t}\right)}$ no existe.

Respecto de esto último, Concluimos que f no puede ser diff en $(0,0)$, pues si lo fuera, entonces por regla de la Cadena

$$(f\circ g)'(0) = Df(0,0)\,g(0)$$

Pero $(f\circ g)'(0)$ no existe.

Concluimos que f no es diff en $(0,0)$

$\boxed{\textbf{P2}}$ **[C2 2010-1 - Correa]**
Considere las funciones $f:\mathbb{R}^3 \to \mathbb{R}^2, g:\mathbb{R}^2 \to \mathbb{R}, h: U \subset \mathbb{R}^2 \to \mathbb{R}$ definidas por

$$f(x,y,z) = (x\sin(z), y\cos(z)), \quad g(x,y) = y^2\cos(x)$$

$$h(x,y) = -y\ln(\cos(x))$$

A partir de estas funciones, se define $\varphi : V \subset \mathbb{R}^2 \to \mathbb{R}^2$ por

$$\varphi(x,y) = f(x+y, g(x,y), h(y,x)).$$

Calcule $D\varphi(x,y)$ y en particular concluya que

$$D\varphi(0,1) = \begin{bmatrix} -\ln(\cos(1)) & 0 \\ 0 & 2 \end{bmatrix}$$

Dem: Buscamos resolver usando regla de la Cadena

Consideremos $\gamma(x,y) = (x+y, g(x,y), h(y,x))$

$$= (\underbrace{[1\ 1](x,y)}_{l_1(x,y)}, g(x,y), h(\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}(x,y)}_{l_2(x,y)}))$$

$$= (\underbrace{l_1(x,y)}_{\gamma_1}, \underbrace{g(x,y)}_{\gamma_2}, \underbrace{(h \circ l_2)(x,y)}_{\gamma_3})$$

luego, $\varphi(x,y) = (f \circ \gamma)(x,y)$, encontremos ahora $Df(x,y,z)$ y $D\gamma(x,y)$ para usar la regla de la Cadena

Recordemos que para toda función lineal l en $\mathbb{R}^n$
$Dl(x) = l \ \forall x \in \mathbb{R}^n$.

luego, Calculando $D\gamma(x,y)$

$$D\gamma(x,y) = \begin{bmatrix} D\gamma_1(x,y) \\ D\gamma_2(x,y) \\ D\gamma_3(x,y) \end{bmatrix} = \begin{bmatrix} Dl_1(x,y) \\ Dg(x,y) \\ D(h \circ l_2)(x,y) \end{bmatrix}$$

Entonces,

$$D\gamma(x,y) = \begin{bmatrix} Dl_1(x,y) \\ Dg(x,y) \\ D(h\circ l_2)(x,y) \end{bmatrix} = \begin{bmatrix} l_1 \\ Dg(x,y) \\ Dh(l_2(x,y))Dl_2(x,y) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ \frac{\partial g}{\partial x}(x,y) & \frac{\partial g}{\partial y}(y,x) \\ \multicolumn{2}{c}{Dh(y,x)\,l_2} \end{bmatrix}$$

Recuerdo: $l_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Calculando los datos faltantes

- $\frac{\partial g}{\partial x}(x,y) = \frac{\partial}{\partial x}\left[y^2\cos(x)\right] = -y^2\operatorname{sen}(x)$
- $\frac{\partial g}{\partial y}(x,y) = \frac{\partial}{\partial y}\left[y^2\cos(x)\right] = 2y\cos(x)$
- $\frac{\partial h}{\partial x}(x,y) = \frac{\partial}{\partial x}\left[-y\ln(\cos(x))\right]$

$$= y\frac{\operatorname{sen}x}{\cos x} = y\tan(x)$$

- $\frac{\partial h}{\partial y}(x,y) = \frac{\partial}{\partial y}\left[-y\ln(\cos(x))\right] = -\ln(\cos(x))$

Entonces, $Dh(y,x) = \left(\frac{\partial h}{\partial x}(y,x) \quad \frac{\partial h}{\partial y}(y,x)\right)$

$$= \left(x\tan(y) \quad -\ln(\cos(y))\right)$$

luego $Dh(y,x)\,l_2 = \left(x\tan(y) \quad -\ln(\cos(y))\right)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$= \left(-\ln(\cos(y)) \quad x\tan(y)\right)$$

Por lo Tanto

$$D\gamma(x,y) = \begin{bmatrix} 1 & 1 \\ -y^2\sin(x) & 2y\cos(x) \\ -\ln(\cos(y)) & x\tan(y) \end{bmatrix}$$

Calculamos ahora $Df(x,y,z)$, recordando que

$$f(x,y,z) = (\underbrace{x\sin(z)}_{f_1}, \underbrace{y\cos(z)}_{f_2})$$

$$Df(x,y,z) = \begin{bmatrix} Df_1(x,y,z) \\ Df_2(x,y,z) \end{bmatrix}$$

Calculando,

$$\frac{\partial f_1}{\partial x} = \sin(z), \quad \frac{\partial f_1}{\partial y} = 0, \quad \frac{\partial f_1}{\partial z} = x\cos(z)$$

$$\frac{\partial f_2}{\partial x} = 0, \quad \frac{\partial f_2}{\partial y} = \cos(z), \quad \frac{\partial f_2}{\partial z} = -y\sin(z)$$

Por lo Tanto,

$$Df(x,y,z) = \begin{bmatrix} \sin(z) & 0 & x\cos(z) \\ 0 & \cos(z) & -y\operatorname{sen}(z) \end{bmatrix}$$

luego, por regla de la cadena

$$D\varphi(x,y) = D(f\circ\gamma)(x,y)$$

$$= Df(\gamma(x,y))D\gamma(x,y)$$

$$= Df(x+y, g(x,y), h(y,x))D\gamma(x,y)$$

Finalmente, encontremos $D\varphi(0,1)$, tenemos

$$D\varphi(0,1) = Df(1,1,0)D\gamma(0,1)$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ -\ln(\cos(1)) & 0 \end{bmatrix}$$

Luego, $D\varphi(0,1) = \begin{bmatrix} -\ln(\cos(1)) & 0 \\ 0 & 2 \end{bmatrix}$

P3 **[C2 2015-2 - Soto]**
Sea $u : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}$ una función de clase $\mathcal{C}^2$ y sea $v : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}$ definida por

$$v(x,y) = u\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right).$$

Decimos que u es armónica si satisface la *ecuación de Laplace* en su dominio, en este caso

$$\Delta u = 0 \text{ en } \mathbb{R}^2 \backslash (0,0)$$

donde $\Delta u = \frac{\partial u^2}{\partial x^2} + \frac{\partial u^2}{\partial y^2}$. Demuestre que u es armónica en $\mathbb{R}^2 \backslash \{(0,0)\}$ sí y solamente sí v es armónica en $\mathbb{R}^2 \backslash \{(0,0)\}$.

Regla de la Cadena: $\boxed{F = g \circ f} \Rightarrow \frac{\partial F_i}{\partial x_j}(x) = \sum_{k=1}^{m} \frac{\partial g_i}{\partial y_k}(f(x)) \frac{\partial f_k}{\partial x_j}(x)$

Dem: Consideramos $\varphi(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$

Entonces $v(x,y) = (u \circ \varphi)(x,y)$, luego

$$\frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial x} \circ \varphi\right) \frac{\partial \varphi_1}{\partial x} + \frac{\partial u}{\partial y} \circ \varphi \cdot \frac{\partial \varphi_2}{\partial x}$$

$$\frac{\partial v}{\partial y} = \left(\frac{\partial u}{\partial x} \circ \varphi\right) \frac{\partial \varphi_1}{\partial y} + \left(\frac{\partial u}{\partial y} \circ \varphi\right) \frac{\partial \varphi_2}{\partial y}$$

luego

$$
\begin{aligned}
\frac{\partial^2 v}{\partial x^2} &= \frac{\partial}{\partial x}\left[\left(\frac{\partial u}{\partial x}\circ \varphi\right)\frac{\partial \varphi_1}{\partial x} + \frac{\partial u}{\partial y}\circ\varphi \cdot \frac{\partial \varphi_2}{\partial x}\right] \\
&= \left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\circ\varphi\right)\frac{\partial \varphi_1}{\partial x} + \left(\frac{\partial u}{\partial x}\circ\varphi\right)\frac{\partial^2 \varphi_1}{\partial x^2}\right) \\
&\quad + \left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\circ\varphi\right)\frac{\partial \varphi_2}{\partial x} + \left(\frac{\partial u}{\partial y}\circ\varphi\right)\frac{\partial^2 \varphi_2}{\partial x^2}\right) \\
&= \left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\circ\varphi\,\frac{\partial \varphi_1}{\partial x} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\frac{\partial \varphi_2}{\partial x}\right)\frac{\partial \varphi_1}{\partial x} + \left(\frac{\partial u}{\partial x}\circ\varphi\right)\frac{\partial^2 \varphi_1}{\partial x^2} \\
&\quad + \left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\circ\varphi\,\frac{\partial \varphi_1}{\partial x} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\frac{\partial \varphi_2}{\partial x}\right)\frac{\partial \varphi_2}{\partial x} + \left(\frac{\partial u}{\partial y}\circ\varphi\right)\frac{\partial^2 \varphi_2}{\partial x^2} \\
&= \left(\frac{\partial^2 u}{\partial x^2}\circ\varphi\cdot\frac{\partial \varphi_1}{\partial x} + \frac{\partial^2 u}{\partial y\partial x}\circ\varphi\cdot\frac{\partial \varphi_2}{\partial x}\right)\frac{\partial \varphi_1}{\partial x} + \left(\frac{\partial u}{\partial x}\circ\varphi\right)\frac{\partial^2 \varphi_1}{\partial x^2} \\
&\quad + \left(\left(\frac{\partial^2 u}{\partial x\partial y}\circ\varphi\right)\frac{\partial \varphi_1}{\partial x} + \left(\frac{\partial^2 u}{\partial y^2}\circ\varphi\right)\cdot\frac{\partial \varphi_2}{\partial x}\right)\frac{\partial \varphi_2}{\partial x} + \left(\frac{\partial u}{\partial y}\circ\varphi\right)\frac{\partial^2 \varphi_2}{\partial x^2} \\
&= \left(\frac{\partial^2 u}{\partial x^2}\circ\varphi\right)\left(\frac{\partial \varphi_1}{\partial x}\right)^2 + \left(\frac{\partial^2 u}{\partial y\partial x}\circ\varphi\right)\left(\frac{\partial \varphi_1}{\partial x}\frac{\partial \varphi_2}{\partial x}\right) + \left(\frac{\partial u}{\partial x}\circ\varphi\right)\frac{\partial^2 \varphi_1}{\partial x^2} \\
&\quad + \left(\frac{\partial^2 u}{\partial x\partial y}\circ\varphi\right)\left(\frac{\partial \varphi_1}{\partial x}\frac{\partial \varphi_2}{\partial x}\right) + \left(\frac{\partial^2 u}{\partial y^2}\circ\varphi\right)\left(\frac{\partial \varphi_2}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\circ\varphi\right)\frac{\partial^2 \varphi_2}{\partial x^2}
\end{aligned}
$$

Por lo tanto,

$$\frac{\partial^2 w}{\partial x^2} = \left(\frac{\partial^2 u}{\partial x^2}\circ\varphi\right)\left(\frac{\partial \varphi_1}{\partial x}\right)^2 + 2\left(\frac{\partial^2 u}{\partial x \partial y}\circ\varphi\right)\left(\frac{\partial \varphi_1}{\partial x}\frac{\partial \varphi_2}{\partial x}\right)$$

$$+ \left(\frac{\partial u}{\partial x}\circ\varphi\right)\frac{\partial^2 \varphi_1}{\partial x^2} + \left(\frac{\partial^2 u}{\partial y^2}\circ\varphi\right)\left(\frac{\partial \varphi_2}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\circ\varphi\right)\frac{\partial^2 \varphi_2}{\partial x^2}$$

Análogamente,

$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial x}\circ\varphi\right)\frac{\partial \varphi_1}{\partial y} + \left(\frac{\partial u}{\partial y}\circ\varphi\right)\frac{\partial \varphi_2}{\partial y}\right]$$

$$= \left(\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\circ\varphi\right)\frac{\partial \varphi_1}{\partial y} + \left(\frac{\partial u}{\partial x}\circ\varphi\right)\frac{\partial^2 \varphi_1}{\partial y^2}\right)$$

$$+ \left(\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\circ\varphi\right)\frac{\partial \varphi_2}{\partial y} + \left(\frac{\partial u}{\partial y}\circ\varphi\right)\frac{\partial^2 \varphi_2}{\partial y^2}\right)$$

$$= \left(\left(\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\right)\circ\varphi\cdot\frac{\partial \varphi_1}{\partial y} + \left(\frac{\partial}{\partial y}\frac{\partial u}{\partial x}\right)\circ\varphi\cdot\frac{\partial \varphi_2}{\partial y}\right)\frac{\partial \varphi_1}{\partial y} + \left(\frac{\partial u}{\partial x}\circ\varphi\right)\frac{\partial^2 \varphi_1}{\partial y^2}\right)$$

$$+ \left(\left(\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial y}\right)\circ\varphi\cdot\frac{\partial \varphi_1}{\partial y} + \left(\frac{\partial}{\partial y}\frac{\partial u}{\partial y}\right)\circ\varphi\cdot\frac{\partial \varphi_2}{\partial y}\right)\frac{\partial \varphi_2}{\partial y} + \left(\frac{\partial u}{\partial y}\circ\varphi\right)\frac{\partial^2 \varphi_2}{\partial y^2}\right)$$

$$= \frac{\partial^2 u}{\partial x^2}\circ\varphi\cdot\left(\frac{\partial \varphi_1}{\partial y}\right)^2 + \frac{\partial^2 u}{\partial y \partial x}\cdot\frac{\partial \varphi_2}{\partial y}\frac{\partial \varphi_1}{\partial y} + \left(\frac{\partial u}{\partial x}\circ\varphi\right)\frac{\partial^2 \varphi_1}{\partial y^2}$$

$$+ \frac{\partial^2 u}{\partial x \partial y}\circ\varphi\,\frac{\partial \varphi_1}{\partial y}\frac{\partial \varphi_2}{\partial y} + \frac{\partial^2 u}{\partial y^2}\cdot\left(\frac{\partial \varphi_2}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y}\circ\varphi\right)\left(\frac{\partial^2 \varphi_2}{\partial y^2}\right)$$

Luego

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x^2}\circ\varphi\cdot\left(\frac{\partial \varphi_1}{\partial y}\right)^2 + 2\underline{\frac{\partial^2 u}{\partial x \partial y}\circ\varphi\cdot\frac{\partial\varphi_1}{\partial y}\frac{\partial\varphi_2}{\partial y}}$$

$$+\left(\frac{\partial u}{\partial x}\circ\varphi\right)\frac{\partial^2\varphi_1}{\partial y^2} + \frac{\partial^2 u}{\partial y^2}\circ\varphi\cdot\left(\frac{\partial\varphi_2}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y}\circ\varphi\right)\left(\frac{\partial^2\varphi_2}{\partial y^2}\right)$$

Por lo tanto $\Delta v = \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}$

$$= \left(\frac{\partial^2 u}{\partial x}\circ\varphi\right)\left(\left(\frac{\partial\varphi_1}{\partial x}\right)^2 + \left(\frac{\partial\varphi_1}{\partial y}\right)^2\right)$$

$$+\left(\frac{\partial^2 u}{\partial y^2}\circ\varphi\right)\cdot\left(\left(\frac{\partial\varphi_2}{\partial x}\right)^2 + \left(\frac{\partial\varphi_2}{\partial y}\right)^2\right)$$

$$+2\frac{\partial^2 u}{\partial y\partial x}\circ\varphi\cdot\left(\frac{\partial\varphi_1}{\partial x}\frac{\partial\varphi_2}{\partial x} + \frac{\partial\varphi_1}{\partial y}\frac{\partial\varphi_2}{\partial y}\right)$$

$$+\left(\frac{\partial u}{\partial x}\circ\varphi\right)\left(\frac{\partial^2\varphi_1}{\partial x^2} + \frac{\partial^2\varphi_1}{\partial y^2}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial^2\varphi_2}{\partial x^2} + \frac{\partial^2\varphi_2}{\partial y^2}\right)$$

Por otro lado, $\dfrac{\partial\varphi_1}{\partial x} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$, $\dfrac{\partial\varphi_1}{\partial y} = \dfrac{-2xy}{(x^2+y^2)^2}$

$$\frac{\partial^2\varphi_1}{\partial y\partial x} = \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}, \quad \frac{\partial^2\varphi_1}{\partial x^2} = \frac{2x(x^2-3y^2)}{(x^2+y^2)^3}$$

$$\frac{\partial^2\varphi_1}{\partial y^2} = \frac{2x(3y^2-x^2)}{(x^2+y^2)^3}$$

Y también $\dfrac{\partial \varphi_2}{\partial x} = \dfrac{-2xy}{(x^2+y^2)^2}$, $\dfrac{\partial \varphi_2}{\partial y} = \dfrac{x^2-y^2}{(x^2+y^2)^2}$

$$\frac{\partial^2 \varphi_2}{\partial x \partial y} = \frac{2x(3y^2-x^2)}{(x^2+y^2)^3} \quad , \quad \frac{\partial^2 \varphi_2}{\partial x^2} = \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}$$

$$\frac{\partial^2 \varphi_2}{\partial y^2} = \frac{2y(y-3x^2)}{(x^2+y^2)^3}$$

Luego

$$\Delta v = \left(\frac{\partial^2 u}{\partial x} \circ \varphi\right)\left(\left(\frac{\partial \varphi_1}{\partial x}\right)^2 + \left(\frac{\partial \varphi_1}{\partial y}\right)^2\right)$$

$$+ \left(\frac{\partial^2 u}{\partial y^2} \circ \varphi\right) \cdot \left(\left(\frac{\partial \varphi_2}{\partial x}\right)^2 + \left(\frac{\partial \varphi_2}{\partial y}\right)^2\right)$$

$$\cancel{+ 2 \frac{\partial^2 u}{\partial y \partial x} \circ \varphi \cdot \left(\frac{\partial \varphi_1}{\partial x}\frac{\partial \varphi_2}{\partial x} + \frac{\partial \varphi_1}{\partial y}\frac{\partial \varphi_2}{\partial y}\right)}$$

$$+ \left(\frac{\partial u}{\partial x} \circ \varphi\right)\cancel{\left(\frac{\partial^2 \varphi_1}{\partial x^2} + \frac{\partial^2 \varphi_1}{\partial y^2}\right)} + \frac{\partial u}{\partial y}\cancel{\left(\frac{\partial^2 \varphi_2}{\partial x^2} + \frac{\partial^2 \varphi_2^2}{\partial y^2}\right)}$$

$$= \left(\underbrace{\left(\frac{y^2-x^2}{(x^2+y^2)^2}\right)^2 + \left(\frac{-2xy}{(x^2+y^2)^2}\right)^2}_{\psi(x,y) > 0}\right) \Delta u \circ \varphi$$

Finalmente, obtenemos que

$$\Delta v = \underbrace{\Psi(x,y)}_{>0} \Delta u \circ \varphi$$

Por lo tanto, $\Delta u = 0 \Rightarrow \Delta v = 0$, por otro lado,

$$\Delta v = 0 \Rightarrow \Delta u(\varphi(x,y)) = 0 \quad \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$$

Puesto que $\varphi(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) = \frac{(x,y)}{\|(x,y)\|^2}$, notamos que $\varphi(\mathbb{R}^2) = \mathbb{R}^2$ pues

$$\|\varphi(x,y)\| = \frac{\|(x,y)\|}{\|(x,y)\|^2} = \frac{1}{\|(x,y)\|}$$

luego podemos armar cualquier dirección y reescalar su norma

Entonces

$$\Delta u(\varphi(x,y)) = 0 \quad \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$$

$$\Rightarrow \Delta u = 0 \quad \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$$

Por lo tanto

$$\underline{\Delta u = 0 \text{ en } \mathbb{R}^2 \setminus \{(0,0)\} \iff \Delta v = 0 \text{ en } \mathbb{R}^2 \setminus \{(0,0)\}}$$

P4 **[Ecuación de Ondas] [C2 2016-1 - Del Pino]**
Suponga que $\varphi : \mathbb{R}^2 \to \mathbb{R}, \varphi = \varphi(x,t)$ es una función de clase C^2 en $\mathbb{R}^2$ que para $c > 0$ satisface

$$\frac{\partial^2 \varphi}{\partial t^2} = c^2 \frac{\partial^2 \varphi}{\partial x^2} \quad \forall (x,t) \in \mathbb{R}^2 \tag{1}$$

Definimos la función auxiliar $\psi(u,v) = \varphi\left(\frac{u+v}{2}, \frac{u-v}{2c}\right)$, buscamos resolver (1), para ello se proponen los siguientes pasos

(a) Demuestre que
$$\frac{\partial^2 \psi}{\partial u \partial v}(u,v) = 0 \quad \forall (u,v) \in \mathbb{R}^2.$$

(b) Deduzca que existen $f, g : \mathbb{R} \to \mathbb{R}$ de clase C^2 tales que
$$\varphi(x,t) = f(x+ct) + g(x-ct).$$

(c) Suponiendo además que $\varphi(x,0) = \varphi_0(x)$ y que $\frac{\partial \varphi}{\partial t}(x,0) = \varphi_1(x)$, muestre que
$$\varphi(x,t) = \frac{1}{2}\left[\varphi_0(x+ct) + \varphi_0(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} \varphi_1(s)ds\right]$$

(a) Consideramos $x(u,v) = \dfrac{u+v}{2}$, $t(u,v) = \dfrac{u-v}{2c}$

luego

$$\frac{\partial \psi}{\partial u} = \frac{\partial \varphi}{\partial x}(x,y)\frac{\partial x}{\partial u} + \frac{\partial \varphi}{\partial y}(x,y)\frac{\partial x}{\partial u}$$

$$= \frac{1}{2}\frac{\partial \varphi}{\partial x}(x,y) + \frac{1}{2c}\frac{\partial \varphi}{\partial y}(x,y)$$

luego

$$\frac{\partial^2 \psi}{\partial v \partial u} = \frac{\partial}{\partial v}\left[\frac{1}{2}\frac{\partial \varphi}{\partial x}(x,y) + \frac{1}{2c}\frac{\partial \varphi}{\partial y}(x,y)\right]$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial x}\right)(x,t) \cdot \frac{\partial x}{\partial v} + \frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial x}\right)(x,t) \cdot \frac{\partial y}{\partial v}\right)$$

$$+ \frac{1}{2c}\left(\frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial t}\right)(x,t)\frac{\partial x}{\partial v} + \frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial t}\right)(x,y) \cdot \frac{\partial y}{\partial v}\right)$$

De lo anterior,

$$\frac{\partial^2 \psi}{\partial v \partial u} = \frac{1}{2}\left(\frac{1}{2}\frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{2c}\frac{\partial^2 \varphi}{\partial t \partial x}\right) + \frac{1}{2c}\left(\frac{1}{2}\frac{\partial^2 \varphi}{\partial x \partial t} - \frac{1}{2c}\frac{\partial^2 \varphi}{\partial t^2}\right)$$

$$= \frac{1}{4}\frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{4c^2}\frac{\partial^2 \varphi}{\partial t^2} = \frac{1}{4}\left(\underbrace{\frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2}}_{0}\right)$$

Usando que φ satisface la ecuación de ondas, deducimos que

$$\boxed{\frac{\partial^2 \psi}{\partial v \partial u} = 0}$$

(b) Puesto que $\frac{\partial}{\partial v}\left[\frac{\partial \psi}{\partial u}\right] = 0$, $\frac{\partial \psi}{\partial u}$ no depende de v

$$\Rightarrow \frac{\partial \psi}{\partial u}(u,v) = \tilde{f}(u) \rightsquigarrow \left(\begin{array}{l}\tilde{f} \in C^1 \text{ pues} \\ \psi \in C^2\end{array}\right)$$

Sea f una primitiva de $\tilde{f}$, luego de igual forma

$$\frac{\partial}{\partial u}\left[\psi - f\right] = 0 \quad (f \in C^2)$$

$$\Rightarrow \underbrace{\psi(u,v)}_{\in C^2} - \underbrace{f(u)}_{\in C^2} = g(v) \quad (\Rightarrow g \in C^2)$$

$$\Rightarrow \psi(u,v) = f(u) + g(v)$$

Como $x(u,v) = \frac{u+v}{2}$, $t(u,v) = \frac{u-v}{2c}$

$$\Rightarrow u = x+ct, \quad v = x-ct$$

Y puesto que $\Psi(u,v) = \varphi(x,t)$, deducimos que existen $f, g : \mathbb{R} \to \mathbb{R}$ de clase C^2 tales que

$$\underline{\varphi(x,t) = f(x+ct) + g(x-ct)}$$

(c) Suponiendo $\varphi(x,0) = \varphi_0(x)$, $\frac{\partial \varphi}{\partial t}(x,0) = \varphi_1(x)$

Tenemos

$$\varphi(x,0) = f(x) + g(x) = \varphi_0(x)$$

$$\frac{\partial}{\partial t}\varphi(x,0) = \left(cf'(x+ct) - cg'(x-ct)\right)\Big|_{t=0}$$

$$= c\left(f'(x) - g'(x)\right) = \varphi_1(x)$$

$$\Rightarrow f(x) - g(x) = \frac{1}{c}\int_0^x \varphi_1(s)\,ds + \underbrace{K}_{\in\mathbb{R}}$$

Tenemos

$$\begin{cases} f(x) + g(x) = \varphi_0(x) \\ f(x) - g(x) = \frac{1}{c}\int_0^x \varphi_1(s)\,ds + K \end{cases}$$

$$\Rightarrow \begin{cases} f(x) = \frac{1}{2}\varphi_0(x) + \frac{1}{2c}\int_0^x \varphi_1(s)\,ds + \frac{K}{2} \\ g(x) = \frac{1}{2}\varphi_0(x) - \frac{1}{2c}\int_0^x \varphi_1(s)\,ds - \frac{K}{2} \end{cases}$$

Por lo tanto, puesto que

$$\varphi(x,t) = f(x+ct) + g(x-ct)$$

Deducimos que

$$\boxed{\varphi(x,t) = \frac{\varphi_0(x+ct) - \varphi_0(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \varphi_1(s)\,ds}$$