Pauta P4 Aux 8

P4 **[Propuesto]** Suponga que $z : \mathbb{R}^2_+ \to \mathbb{R}$ es una función de clase $\mathcal{C}^2$, que satisface la ecuación

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0 \quad \forall (x,y) \in \mathbb{R}^2_+.$$

Considere el cambio de variables definido por $u(x,y) = x+y, v(x,y) = \frac{y}{x}$, y la nueva función $\omega = \omega(u,v)$ definida por $z(x,y) = x\omega(u(x,y), v(x,y))$. Demostrar que la ecuación que satisface la función $\omega(u,v)$ es

$$\frac{(1+v)^3}{u}\frac{\partial^2 \omega}{\partial v^2}(u,v) = 0 \quad \forall (u,v) \in \mathbb{R}^2_+$$

Dem: Tenemos $z : \mathbb{R}^2_+ \to \mathbb{R}$ de clase C^2 tal que

$$\frac{\partial^2 z}{\partial x} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

Y consideremos $u(x,y) = x+y$, $v(x,y) = \frac{y}{x}$ y $w = w(u,v)$ definido por $z(x,y) = x\,w(u(x,y), v(x,y))$

Consideremos $\varphi(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{pmatrix} x+y \\ y/x \end{pmatrix}$, entonces

Objetivo: Encontrar una ec. para w usando la Regla de la Cadena.

•) $$\frac{\partial z}{\partial x}(x,y) = \frac{\partial}{\partial x}\Big[\underbrace{x \cdot (w\circ\varphi)(x,y)}_{\text{Producto}}\Big]$$

$$= (w\circ\varphi)(x,y) + x\frac{\partial}{\partial x}(w\circ\varphi)(x,y) \to \text{Regla de la Cadena}$$

$$= (w\circ\varphi)(x,y) + x\left(\frac{\partial w}{\partial u}(\varphi(x,y))\underbrace{\frac{\partial u}{\partial x}(x,y)}_{1} + \frac{\partial w}{\partial v}(\varphi(x,y))\underbrace{\frac{\partial v}{\partial x}(x,y)}_{-y/x^2}\right)$$

$$= w(u,v) + \left(\frac{\partial w}{\partial u}(u,v) - \frac{y}{x^2}\frac{\partial w}{\partial v}(u,v)\right)$$

•) $$\frac{\partial z}{\partial y}(x,y) = x\frac{\partial}{\partial y}(w\circ\varphi)(x,y)$$

$$= x\left(\frac{\partial w}{\partial u}(u,v)\underbrace{\frac{\partial u}{\partial y}(x,y)}_{1} + \frac{\partial w}{\partial v}(u,v)\underbrace{\frac{\partial v}{\partial y}(x,y)}_{\frac{1}{x}}\right)$$

Por lo tanto tenemos

$$\bullet)\quad \frac{\partial z}{\partial x} = w(u,v) + x\left(\frac{\partial w}{\partial u}(u,v) - \frac{y}{x^2}\frac{\partial w}{\partial v}(u,v)\right)$$

$$\bullet)\quad \frac{\partial z}{\partial y} = x\left(\frac{\partial w}{\partial u}(u,v) + \frac{1}{x}\frac{\partial w}{\partial v}(u,v)\right)$$

luego,

$$\bullet)\quad \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left[w(\varphi(x,y)) + x\left(\frac{\partial w}{\partial u}(\varphi(x,y)) - \frac{y}{x^2}\frac{\partial w}{\partial v}(\varphi(x,y))\right)\right]$$

$$= \left(\frac{\partial w}{\partial u}\underbrace{\frac{\partial u}{\partial x}}_{1} + \frac{\partial w}{\partial v}\underbrace{\frac{\partial v}{\partial x}}_{-y/x^2}\right)$$

$$+ \left(\frac{\partial w}{\partial u}(\varphi(x,y)) - \frac{y}{x^2}\frac{\partial w}{\partial v}(\varphi(x,y))\right)$$

$$+ x\frac{\partial}{\partial x}\left[\frac{\partial w}{\partial u}(\varphi(x,y)) - \frac{y}{x^2}\frac{\partial w}{\partial v}(\varphi(x,y))\right]$$

$$= \left(\frac{\partial w}{\partial u}(u,v) - \frac{y}{x^2}\frac{\partial w}{\partial v}(u,v)\right) + \left(\frac{\partial w}{\partial u}(u,v) - \frac{y}{x^2}\frac{\partial w}{\partial v}(u,v)\right)$$

$$+ x\left[\frac{\partial}{\partial u}\left(\frac{\partial w}{\partial u}\right)\underbrace{\frac{\partial u}{\partial x}}_{1} + \frac{\partial}{\partial v}\left(\frac{\partial w}{\partial u}\right)\underbrace{\frac{\partial v}{\partial x}}_{-y/x^2} + \frac{\partial}{\partial x}\left(-\frac{y}{x^2}\frac{\partial w}{\partial v}\right)\right]$$

Desarrollando

$$\frac{\partial^2 z}{\partial x^2}(x,y) = 2\left(\frac{\partial w}{\partial u}(u,v) - \underbrace{\frac{y}{x^2}\frac{\partial w}{\partial v}(u,v)}\right)$$

$$+ x\left[\frac{\partial^2 w}{\partial u^2}(u,v) - \underbrace{\frac{y}{x^2}\frac{\partial^2 w}{\partial v \partial u}} + \right.$$

$$\left.\left(2\frac{y}{x^3}\frac{\partial w}{\partial v} - \underbrace{\frac{y}{x^2}\left(\frac{\partial^2 w}{\partial u \partial v} - \frac{y}{x^2}\frac{\partial^2 w}{\partial v^2}\right)}\right)\right]$$

Por otro lado,

$$\frac{\partial^2 z}{\partial y^2}(x,y) = \frac{\partial}{\partial y}\left[x\left(\frac{\partial w}{\partial u}(u,v) + \frac{1}{x}\frac{\partial w}{\partial v}(u,v)\right)\right]$$

$$= x\left[\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial u}(u,v)\right) + \frac{1}{x}\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial v}(u,v)\right)\right]$$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial w}{\partial u}\right)\underbrace{\frac{\partial u}{\partial y}}_{1} + \frac{\partial}{\partial v}\left(\frac{\partial w}{\partial u}\right)\underbrace{\frac{\partial v}{\partial y}}_{1/x}\right.$$

$$\left. + \frac{1}{x}\left(\frac{\partial}{\partial u}\left(\frac{\partial w}{\partial v}\right)\underbrace{\frac{\partial u}{\partial y}}_{1} + \frac{\partial}{\partial v}\left(\frac{\partial w}{\partial v}\right)\underbrace{\frac{\partial v}{\partial y}}_{1/x}\right)\right]$$

$$= x\frac{\partial^2 w}{\partial u^2}(u,v) + \left(\frac{\partial^2 w}{\partial v \partial u} + \frac{\partial^2 w}{\partial u \partial v}\right) + \frac{1}{x}\frac{\partial^2 w}{\partial v^2}$$

Por lema de Schwartz, como w es de clase C^2,

$$\frac{\partial^2 w}{\partial v \partial u} = \frac{\partial^2 w}{\partial u \partial v} \Rightarrow \frac{\partial^2 w}{\partial v \partial u} + \frac{\partial^2 w}{\partial u \partial v} = 2\frac{\partial^2 w}{\partial u \partial v}$$

Y por otra parte,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left[x\left(\frac{\partial w}{\partial u}(u,v) + \frac{1}{x}\frac{\partial w}{\partial v}(u,v)\right)\right]$$

$$= \left(\frac{\partial w}{\partial u}(u,v)+\frac{1}{x}\frac{\partial w}{\partial v}(u,v)\right)$$

$$+ x\left[\frac{\partial}{\partial x}\left[\frac{\partial w}{\partial u}(u,v)+\frac{1}{x}\frac{\partial w}{\partial v}(u,v)\right]\right]$$

$$= \left(\frac{\partial w}{\partial u}(u,v)+\frac{1}{x}\frac{\partial w}{\partial v}(u,v)\right)$$

$$+ x\left[\frac{\partial}{\partial u}\left(\frac{\partial w}{\partial u}\right)\underbrace{\frac{\partial u}{\partial x}}_{1} + \frac{\partial}{\partial v}\left(\frac{\partial w}{\partial u}\right)\underbrace{\frac{\partial v}{\partial x}}_{-y/x^2} + \underbrace{\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial w}{\partial v}(u,v)\right)}_{-\frac{1}{x^2}\frac{\partial w}{\partial v} + \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial v}\right)\overset{1}{\frac{\partial u}{\partial x}} + \frac{\partial}{\partial v}\left(\frac{\partial w}{\partial v}\right)\underbrace{\frac{\partial v}{\partial x}}_{-y/x^2}}\right]$$

Simplificando

$$\frac{\partial^2 z}{\partial x \partial y} = \left(\frac{\partial w}{\partial u}(u,v)+\frac{1}{x}\frac{\partial w}{\partial v}(u,v)\right)$$

$$+ x\left(\frac{\partial^2 w}{\partial u^2} + \frac{1}{x}\frac{\partial^2 w}{\partial u \partial v} - \frac{1}{x^2}\frac{\partial w}{\partial v} - \frac{y}{x^2}\left(\frac{\partial^2 w}{\partial u \partial v} + \frac{1}{x}\frac{\partial^2 w}{\partial v^2}\right)\right)$$

Luego, puesto que

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0 \quad \forall (x,y) \in \mathbb{R}^2_+$$

$$\Rightarrow \Big[2\left(\frac{\partial w}{\partial u}(u,v) - \frac{y}{x^2}\frac{\partial w}{\partial v}(u,v) \right) \quad \Big\} \frac{\partial^2 z}{\partial x^2}$$

$$+ x \Big[\frac{\partial^2 w}{\partial u^2}(u,v) - \underbrace{\frac{y}{x^2}\frac{\partial^2 w}{\partial v \partial u}} +$$

$$\left(2\frac{y}{x^3}\frac{\partial w}{\partial v} - \frac{y}{x^2}\left(\frac{\partial^2 w}{\partial u \partial v} - \frac{y}{x^2}\frac{\partial^2 w}{\partial v^2} \right) \right) \Big]$$

$$-2\Big[\left(\frac{\partial w}{\partial u}(u,v) + \frac{1}{x}\frac{\partial w}{\partial v}(u,v) \right) \quad \Big\} -2\frac{\partial^2 z}{\partial x \partial y}$$

$$+ x\left(\frac{\partial^2 w}{\partial u^2} + \frac{1}{x}\frac{\partial^2 w}{\partial u \partial v} - \frac{1}{x^2}\frac{\partial w}{\partial v} - \frac{y}{x^2}\left(\frac{\partial^2 w}{\partial u \partial v} + \frac{1}{x}\frac{\partial^2 w}{\partial v^2} \right) \right) \Big]$$

$$+ \left(x\frac{\partial^2 w}{\partial u^2}(u,v) + 2\frac{\partial^2 w}{\partial v \partial u} + \frac{1}{x}\frac{\partial^2 w}{\partial v^2} \right) \Big] \Big\} \frac{\partial^2 z}{\partial y^2}$$

$$= 0$$

Simplificando tenemos

$$\frac{\partial w}{\partial u}(2-2) + \frac{\partial w}{\partial v}\Big(\underbrace{-\frac{2y}{x^2} + \frac{2y}{x^2} - \frac{2}{x} + \frac{2}{x}}_{=0} \Big) + \frac{\partial^2 w}{\partial u^2}(\underbrace{x - 2x + x}_{=0})$$

$$+ \frac{\partial^2 w}{\partial v^2}\Big(\underbrace{\frac{y^2}{x^3} + \frac{2y}{x^2} + \frac{1}{x}}_{\frac{1}{x^3}(\underbrace{y^2 + 2yx + x^2}_{(x+y)^2})} \Big) + \frac{\partial^2 w}{\partial u \partial v}\Big(\underbrace{-\frac{y}{x} - \frac{y}{x} + \frac{2y}{x} - 2 + 2}_{=0} \Big) = 0$$

Tenemos entonces

$$\frac{(x+y)^2}{x^3}\frac{\partial^2 w}{\partial u \partial v}(u(x,y),v(x,y)) = 0 \quad \forall (x,y)\in \mathbb{R}^2_+$$

Usando que $u(x,y) = x+y$, $v(x,y) = \frac{y}{x} \Rightarrow y = xv$

$$\Rightarrow u = x + xv \Rightarrow u = x(1+v)$$

$$\Rightarrow x = \frac{u}{1+v}$$

Luego,

$$\frac{(x+y)^2}{x^3} = \cancel{u^2}\cdot\frac{(1+v)^3}{\cancel{u^3}} = \frac{(1+v)^3}{u}$$

Entonces concluimos usando que $u(\mathbb{R}^2_+) = \mathbb{R}_+$, $v(\mathbb{R}^2_+) = \mathbb{R}_+$ que se cumple la ecuación

$$\boxed{\frac{(1+v)^3}{u}\frac{\partial^2 w}{\partial v^2}(u,v) = 0 \quad \forall (u,v)\in \mathbb{R}^2_+}$$