Pauta PP S13 — 27/06

1. Encuentren la forma binomial, la forma polar y ubiquen en el plano los siguientes números complejos:

   (a) i^{-1}.

   (b) $\frac{\overline{2+i}}{1+3i^3}$.

   (c) $-2e^{i\frac{\pi}{3}}$.

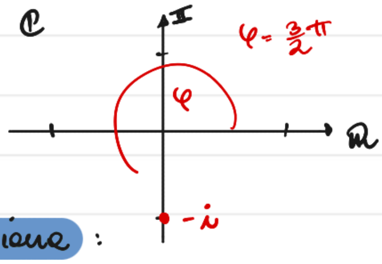
(a) Forma binomial / Cartesiana:

(NIKITA)

$$i^{-1} = \frac{1}{i} = \frac{i}{i^2} = -i = 0 - i$$

Forma Polar: $|z| = \sqrt{0^2+(-1)^2} = 1$, $\arg(-i) = \frac{3}{2}\pi$

$$-i = |-i|\,e^{i\arg(-i)} = e^{i\frac{3}{2}\pi}$$



(b) Forma binomial / Cartesiana:

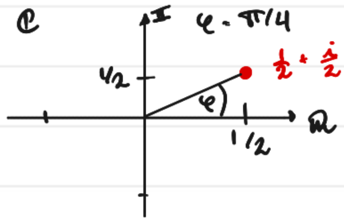
$$\frac{\overline{2+i}}{1+3i^3} = \frac{2-i}{1-3i}\cdot\frac{1+3i}{1+3i} = \frac{5+5i}{10} = \frac{1}{2}+\frac{1}{2}i$$

Primer Cuadrante ☆

Forma Polar: $|z| = \sqrt{\frac{1}{4}+\frac{1}{4}} = \frac{1}{\sqrt{2}}$,

$$\arg\left(\frac{1}{2}+\frac{i}{2}\right) = \pi/4$$

$$\Rightarrow \frac{1}{2}+\frac{i}{2} = \frac{1}{\sqrt{2}}e^{i\pi/4}$$



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(c) Forma Polar: $-2e^{i\frac{\pi}{3}} = 2e^{i\pi}e^{i\pi/3} = 2e^{i(\pi+\pi/3)}$

Forma Binomial / Cartesiana:

$$-2e^{i\frac{\pi}{3}} = -2(\underbrace{\cos(\pi/3)}_{1/2} + i\underbrace{\sin(\pi/3)}_{\sqrt{3}/2}) = -1-\sqrt{3}i$$

2. (a) Prueben que si $z = |z|e^{i\theta}$, entonces $\overline{z} = |z|e^{-i\theta}$ y $z^{-1} = \frac{1}{|z|}e^{-i\theta}$ si $|z| \neq 0$.

(b) Encuentren todos los complejos $z \neq 0$ tales que $z^{-1} = \overline{z}$.

Dem (a) Usando que $e^{i\theta} = \cos\theta + i \operatorname{sen}\theta$ y $\operatorname{sen}(-\theta) = -\operatorname{sen}(\theta)$, puesto que

$$z = |z| e^{i\theta} = |z| \left(\cos(\theta) + i \operatorname{sen}(\theta) \right)$$

$$\Rightarrow \overline{z} = |z| \left(\cos(\theta) - i \operatorname{sen}(\theta) \right)$$

$$= |z| \left(\cos(\theta) + i \operatorname{sen}(-\theta) \right) = |z| e^{-i\theta}$$

$$\Rightarrow \overline{z} = |z| e^{-i\theta}$$

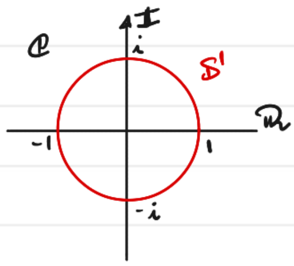
Por otro lado,

$$z^{-1} = \left(|z| e^{i\theta} \right)^{-1} = \frac{1}{|z|} e^{-i\theta} \quad \forall z \neq 0$$

(b) Supongamos $z \in \mathbb{C} \setminus \{0\}$ tal que $z^{-1} = \overline{z}$

$$\Leftrightarrow z z^{-1} = z \overline{z} \Leftrightarrow |z|^2 = 1$$

$$\Leftrightarrow |z| = 1 \Leftrightarrow z \in S^1$$



Equivalentemente, los complejos de módulo igual a 1.

3. Resuelvan la ecuación $z^4 = 2 - i\sqrt{3}$ en $\mathbb{C}$.

(P3) Vemos que $2 - i\sqrt{3} = \sqrt{4+3}\, e^{i\arg(2-i\sqrt{3})}$

Converting between polar and Cartesian coordinates [edit]

The polar coordinates r and φ can be converted to the Cartesian coordinates x and y by using the trigonometric functions sine and cosine:

$$x = r\cos\varphi,$$
$$y = r\sin\varphi.$$

The Cartesian coordinates x and y can be converted to polar coordinates r and φ with $r \geq 0$ and φ in the interval $(-\pi, \pi]$ by:[13]

$$r = \sqrt{x^2+y^2} = \operatorname{hypot}(x,y)$$
$$\varphi = \operatorname{atan2}(y,x),$$

where hypot is the Pythagorean sum and atan2 is a common variation on the arctangent function defined as

$$\operatorname{atan2}(y,x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0 \\ \arctan\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan\left(\frac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

Tenemos

$$x > 0,\ y < 0$$

$$\Rightarrow \varphi = \arctan\left(\frac{y}{x}\right)$$

Luego,

$$\varphi = \arg(2 - i\sqrt{3}) = \arctan\left(\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow 2 - i\sqrt{3} = \sqrt{7}\, e^{i\arctan\left(\frac{\sqrt{3}}{2}\right)}$$

$$= \sqrt{7}\, e^{i\arctan\left(\frac{\sqrt{3}}{2}\right)} + 2k\pi \quad \forall k \in \mathbb{N}$$

Entonces tenemos,

$$z^4 = 2 - i\sqrt{3} = \sqrt{7}\, e^{i\arctan\left(\frac{\sqrt{3}}{2}\right)} + 2k\pi$$

$$\Rightarrow z = \left(\sqrt{7}\, e^{i\left(\arctan\left(\frac{\sqrt{3}}{2}\right) + 2k\pi\right)}\right)^{1/4} \quad \forall k \in \mathbb{N}$$

$$= \sqrt[8]{7}\, e^{i\left(\arctan\left(\frac{\sqrt{3}}{2}\right)/4 + \frac{k\pi}{2}\right)} \quad \forall k \in \mathbb{N}$$

Equivalentemente, las soluciones son

$$z = \sqrt[8]{7}\, e^{i\left(\arctan\left(\frac{\sqrt{3}}{2}\right)/4 + \frac{k\pi}{2}\right)} \quad \forall k \in \{0, 1, 2, 3\}$$