$x - y \leq z \Rightarrow \quad \sup(A) - \sup(B) - \sup(C) > 0$

$\sup(A) - \sup(C) > \sup(B) \geq y \geq x - z$

$x - z \leq y$

$x \leq z + y$

$\sup(A) - \varepsilon < x$

$-\sup(A) \leq -z$

$\cancel{\sup(A)} - \cancel{\sup(A)} + \sup(B) + \sup(C) < x \leq z + y$

$\sup(B) < y$

Problemas para Estudio.

$$\lim_{n\to+\infty} \frac{n!}{n^n}$$

Teo Sándwich.

$$b_n \le a_n \le c_n$$

$$b_n \xrightarrow{n\to\infty} l \qquad a_n \longrightarrow l \qquad c_n \xrightarrow{n\to\infty} l$$

$$\lim_{n\to+\infty} \frac{n!}{n^n}$$

$$0 \le \frac{n!}{n^n} = \frac{\overset{\le n}{1}\cdot\overset{\le n}{2}\cdot\overset{\le n}{3}\cdot\overset{\le n}{4}\cdots\overset{\le n}{(n-1)}\cdot n}{\underbrace{n\cdot n\cdot n\cdot n\cdots n}_{n\text{-veces}}} \le \frac{n\cdot n\cdot n\cdot n\cdots \overset{n-1}{n}\cdot n}{n^n}$$

$$\lim_{n \to \infty} \frac{n!}{n^n}$$

$$\begin{array}{ccccc} a_n & \leq & c_n & \leq & b_n \\ \downarrow & & \downarrow & & \downarrow \\ \ell & & \ell & & \ell \end{array}$$

$$0 \leq \frac{n!}{n^n} = \frac{\overset{\leq n}{1} \cdot \overset{\leq n}{2} \cdot \overset{\leq n}{3} \cdots \overset{\leq n}{(n-1)}\, n}{n \cdot n \cdot n \cdot n \cdots n} \leq \frac{n^{n-1}}{n^n} = \frac{1}{n}$$

$$0 \to 0, \quad \frac{n!}{n^n} \to 0, \quad \frac{1}{n} \to 0$$

$$\lim_{n \to \infty} \frac{n!}{n^n} = 0 \qquad \text{TEO Sándwich}$$

$$\lim_{n \to \infty} \sum_{k=1}^{n} \left(\frac{1}{k+n} \right)^2 \qquad \alpha^2 \geq 0$$

$$0 \leq \sum_{k=1}^{n} \left(\frac{1}{k+n} \right)^2 \leq \sum_{k=1}^{n} \frac{1}{n^2} = \frac{1}{n^2} \cdot 1 (n-1+1)$$

$$= \frac{1}{n} \cdot \frac{1}{n}$$

$0 \qquad\qquad 0$

$$\Rightarrow \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{k+n}\right)^2 = 0 \; //$$

Partial

$$\lim_{n\to\infty} \frac{2^n+1}{n^2\cdot 3^{n+1}} = \lim_{n\to\infty} \frac{2^n}{n^2\cdot 3^n}\cdot\frac{1}{3} + \frac{1}{n^2\cdot 3^{n+1}}$$

$$= \lim_{n\to\infty} \underbrace{\cancel{\left(\frac{2}{3}\right)^n}^{\,0}\cdot \cancel{\frac{1}{n}}^{\,0}\cdot\cancel{\frac{1}{n}}^{\,0}\cdot\cancel{\frac{1}{3}}^{\,\frac{1}{3}}}_{\text{nula}\cdot\text{acotado}} + \underbrace{\cancel{\frac{1}{n}}^{\,0}\cdot\cancel{\frac{1}{n}}^{\,0}\cdot\cancel{\left(\frac{1}{3}\right)^n}^{\,0}\cdot\cancel{\frac{1}{3}}^{\,\frac{1}{3}}}_{\text{nula}\cdot\text{acotad}}$$

$q^n,\ |q|<1 \Rightarrow q^n \to 0 \qquad = 0$

•) $\displaystyle \lim_{n\to\infty} \Big(\underbrace{\frac{3n-1}{2n+4}}_{s_n}\Big)^n \qquad \boxed{|q_n|^n}$

$$\frac{3n-1}{2n+4} \to \frac{3}{2}, \quad \lim s_n = \frac{3}{2} > 1$$

$$\lim_{n\to\infty} (S_n)^n = +\infty \quad \text{por aparte}$$

- $$\lim_{n\to\infty} \sqrt[n]{\frac{2^n+1}{n^2\cdot 3^{n+1}}}$$

$$1 \le 2^n$$

$$0 \le \underbrace{\frac{2^n}{n^2 3^{n+1}} \le \frac{2^n+1}{n^2\cdot 3^{n+1}} \le \frac{2^n+2^n}{n^2\cdot 3^{n+1}} = \frac{2^{n+1}}{n^2\cdot 3^{n+1}}} \quad / \sqrt[n]{}$$

$$\sqrt[n]{\frac{2^n}{3^n}\cdot\frac{1}{3}\cdot\frac{1}{n^2}} \le \sqrt[n]{\frac{2^n+1}{n^2\cdot 3^{n+1}}} \le \sqrt[n]{\frac{2^n}{3^n}\cdot\frac{2}{3}\cdot\frac{1}{n^2}}$$

$$\frac{2}{3}\sqrt[n]{\frac{1}{3n^2}} \le \sqrt[n]{\frac{2^n+1}{n^2\cdot 3^{n+1}}} \le \frac{2}{3}\sqrt[n]{\frac{2}{3}\cdot\frac{1}{n^2}}$$

$$\frac{2}{3}\cdot\frac{\cancel{\sqrt[n]{1}}^{\,1}}{\cancel{\sqrt[n]{3}}^{\,1}\cdot\cancel{\sqrt[n]{n^2}}^{\,1}} \le \sqrt[n]{\frac{2^n+1}{n^2\cdot 3^{n+1}}} \le \frac{2}{3}\,\frac{\cancel{\sqrt[n]{2}}^{\,1}}{\cancel{\sqrt[n]{3}}^{\,1}}\cdot\frac{1}{\cancel{\sqrt[n]{n^2}}^{\,1}}$$

$\downarrow$ $\frac{2}{3}$ // $\downarrow$ $\frac{2}{3}$ //

$$\sqrt[n]{a} \to 1,\ a>0$$

$\sqrt[n]{n^2} \to 1$

$\sqrt[n]{n} \cdot \sqrt[n]{n}$

• $\lim \sqrt[n]{n^3 + 100n^2 + 3}$

$$\sqrt[n]{n^3} \le \sqrt[n]{n^3 + 100n^2 + 3} \le \sqrt[n]{n^3 + 100n^3 + 3n^3}$$

$$= \sqrt[n]{104\, n^3}$$

$$= \cancel{\sqrt[n]{104}}^{\,1} \cdot \cancel{\sqrt[n]{n}}^{\,1} \cdot \cancel{\sqrt[n]{n}}^{\,1} \cdot \cancel{\sqrt[n]{n}}^{\,1}$$

$$\left(\cancel{\sqrt[n]{n}}^{\,1}\right)^3 \le$$

$\downarrow$ $\quad\quad\quad\quad\quad\quad\quad$ $\downarrow$

1 $\quad\quad\quad\quad\quad\quad\quad\quad$ 1

TEO Sándwich $\sqrt[n]{n^3 + 100n^2 + 3} \to 1$

$$\lim_{n \to \infty} \sqrt[n]{7^n + 5^n + 3^n}$$

$$\sqrt[n]{7^n} \leq \sqrt[n]{7^n + \underbrace{5^n + 3^n}} \leq \sqrt[n]{7^n + 7^n + 7^n} \leq \cancel{\sqrt[n]{3}}^{\,1} \sqrt[n]{7^n}$$

$$\begin{array}{ccc} \| & & \\ 7 \leq & \downarrow & \downarrow \\ \downarrow & & 1 \cdot 7 \\ 7 & & \end{array}$$

Teo Sandwich tiende a 7 //

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad \Big| \quad \exp(\ln(x)) = x$$

$$m = n^2 + 1$$

$$n \to \infty \qquad m \to \infty$$

$$\cdot) \lim_{n \to \infty} \left(\frac{n^2+1+1}{n^2+1}\right)^{n^2} = \lim_{n \to \infty} \frac{\cancel{\left(1 + \frac{1}{n^2+1}\right)^{n^2+1}} \to e^1}{\cancel{\left(1 + \frac{1}{n^2+1}\right)^{1}}} = e^1 //$$

$a_0 \in (0,1)$

$a_1 \in (0,1)$, $a_{k+1} = a_k \dfrac{(1+a_{k-1}^2)}{2}$

PDQ| $a_n \in (0,1)$, $\forall n \geq 2$

CB| $a_2 = a_1 \dfrac{(1+a_0^2)}{2}$

$$a_0 \in (0,1) \Rightarrow a_0^2 \in (0,1)$$

$$\Rightarrow \frac{a_0^2}{2} \in \left(0,\frac{1}{2}\right) \Rightarrow \frac{a_0^2}{2}+\frac{1}{2} \in \left(\frac{1}{2},1\right)$$

$$0 < \frac{1}{2} < \frac{a_0^2+1}{2} < 1$$

$$0 < \frac{a_0^2+1}{2} < 1 \quad / \cdot a_1 \in (0,1)$$

$$0 \cdot a_1 < a_1 \frac{(a_0^2+1)}{2} < a_1 < 1$$

$$\Rightarrow 0 < a_1 \frac{(a_0^2+1)}{2} < 1$$

$$\parallel$$

$$a_2 \in (0,1)$$

HI| Asumo que $a_{k+1} = a_k \dfrac{(1+a_{k-1}^2)}{2} \in (0,1)$

SPG

$\forall n \leq k+1$, para algún $k+1$ natural.

PI

$$a_{k+2} = a_{k+1}\frac{(1+a_k^2)}{2} \qquad k \leq k+1$$

$a_k \in (0,1)$ HI

$$0 < a_k^2 < 1 \quad / +1$$

$$1 < 1+a_k^2 < 2 \quad / \cdot \frac{1}{2}$$

$$0 < \frac{1}{2} < \frac{1+a_k^2}{2} < 1 \quad /\cdot$$

$(k+1) \leq k+1$

$$0 < \frac{1+a_k^2}{2} < 1 \quad / \cdot a_{k+1} \in (0,1)$$

HI

$$\underbrace{0 \cdot a_{k+1}}_{} < a_{k+1}\frac{(1+a_k^2)}{2} < a_{k+1} < 1$$

$$0 < \underbrace{a_{k+1}\frac{(1+a_k^2)}{2}}_{} < 1$$

$$0 < a_{k+2} < 1$$

PPI Fuerte

Si $\forall n \leq k+1 \ a_n \in (0,1)$, $a_{k+2} \in (0,1)$

Como $a_n \in (0,1)$, $\forall n \in \mathbb{N}$.

$\Rightarrow |a_n| \leq 1 = M \Leftrightarrow$ Acotado

Veamos crecimiento $\qquad a_{n-1}^2 \in (0,1)$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{\cancel{a_n}(a_{n-1}^2+1)}{2}}{\cancel{a_n}} = \frac{a_{n-1}^2+1}{2} < \frac{1+1}{2} = 1$$

$$\frac{a_{n+1}}{a_n} < 1 \Rightarrow \text{decreciente}$$

TSMI $\Rightarrow$ a_n converge, $a_n \to \ell$

$$a_n = \frac{a_{n-1}(1+a_{n-2}^2)}{2} \Big/ \lim_{n\to\infty}$$

$$\ell = \frac{\ell(1+\ell^2)}{2} \Rightarrow 2\ell = \ell + \ell^3$$

$$0 = \ell^3 - \ell$$

$$0 = \ell(\ell^2 - 1$$

$l=0 \vee \cancel{l=1} \vee \cancel{l=-1}$

a_n decrece $\quad a_n \in (0,1)$

$l = 0$ ✓ pues a_n converge y es decreciente

•) Sea sucesión creciente

$u_n \nearrow$

$v_n \downarrow \quad$ Si $\lim(u_n - v_n) = 0$

PDQ: $\lim u_n = \lim v_n = l$

$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0 \; |u_n - v_n - 0| < \varepsilon$

$\varepsilon = 1, \; \exists n_0 \in \mathbb{N}, \; \forall n \geq n_0$

$$|u_n - v_n| < 1$$

$$-1 < u_n - v_n < 1 \quad / + v_n$$

$$-1 + v_n < \underbrace{u_n < 1 + v_n}, \; \exists n_0$$

$$u_n < 1 + v_n < \underbrace{1 + v_{n_0}}_{M}$$

$$u_n < 1 + v_{n_0} = M$$

$\Rightarrow$ u_n acotado sup y sabemos $u_n \nearrow$

$\Rightarrow$ TSM $\exists \lim u_n$

$$-1 < \underbrace{u_n - v_n < 1}$$

$$-1 < u_n - v_n \quad \wedge \quad u_n - v_n < 1$$

$$\underbrace{v_n - u_n < 1} \quad \wedge \quad \underbrace{-1 < v_n - u_n}$$

$$\Rightarrow \quad \underbrace{-1 < v_n - u_n}_{\exists n_0} < 1$$

$$\underbrace{u_{n_0} - 1}_{\bar{m}} < u_n - 1 < v_n$$

$\bar{m} \leq v_n \Rightarrow v_n$ acotado inf y $v_n \searrow$

$\Rightarrow$ TSM converge

Como u_n, v_n convergen usemos

$$\lim (u_n - v_n) = 0$$
$$\lim u_n - \lim v_n = 0$$
$$\lim u_n = \lim v_n \,//$$

$$\int_{\frac{a}{4}}^{\frac{a}{2}} \frac{dx}{ax-x^2} \quad \sqrt{ax-x^2} \cdot \frac{\sqrt{ax-x^2}}{\sqrt{ax-x^2}} + \text{completar cuadrado}$$

$\Rightarrow$ sal arctg o ln ✓

Que tengan Buena semana,
dudas por el foro :)