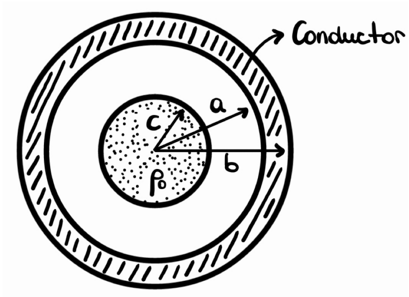
Pauta P3

3. Se tiene un casquete conductor esférico de radio interior a y exterior b. Al centro del casquete existe una distribución de carga uniforme ρ_0 de radio c. Calcule la energía electrostática del sistema.



Buscamos la energía electroestática del sistema: $U = \frac{\varepsilon_0}{2} \iiint_V |\vec{E}|^2 dV$

$\therefore$ Calcularemos $\vec{E}$ en todo el espacio utilizando Ley de Gauss: $\iint_\Omega \vec{E} \cdot \vec{ds} = \frac{Q_{enc}}{\varepsilon_0}$

Nota: Se busca resolver tanto el flujo $\iint \vec{E}\,\vec{ds}$ como la carga encerrada $Q_{enc} = \iiint \rho\, dV$ y con ello obtener una ecuación donde la incognita sea $E(r)$, utilizando la simetría radial del campo.

$0 \le r \le c$ | $\vec{E}_1 = E_1 \hat{r} \Rightarrow \underbrace{\int_0^{2\pi}\int_0^{\pi} E_1 \cancel{\hat{r}} \cdot r^2 \operatorname{sen}(\theta) d\theta d\phi \cancel{\hat{r}}}_{\text{flujo}} = \underbrace{\frac{1}{\varepsilon_0}\int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho_0\, r^2 \operatorname{sen}\theta\, dr\, d\theta\, d\phi}_{\text{carga}}$ $\quad \to$ depende de r

$\Rightarrow \cancel{4\pi r^2} E_1 = \frac{\cancel{4}}{3}\cancel{\pi} r^{\cancel{3}} \rho_0 \cdot \frac{1}{\varepsilon_0}$ $\quad$ 1,0

$\Rightarrow \vec{E}_1 = \frac{r\rho_0}{3\varepsilon_0}\hat{r}$ //

$c \le r \le a$ | $\vec{E}_2 = E_2 \hat{r} \Rightarrow \int_0^{2\pi}\int_0^{\pi} E_2 \cancel{\hat{r}} \cdot r^2 \operatorname{sen}(\theta) d\theta d\phi \cancel{\hat{r}} = \frac{1}{\varepsilon_0}\int_0^{2\pi}\int_0^{\pi}\int_0^{c} \rho_0\, r^2 \operatorname{sen}\theta\, dr\, d\theta\, d\phi$ $\quad \to$ no depende de r

$\Rightarrow \cancel{4\pi} r^2 E_2 = \frac{\cancel{4}}{3}\cancel{\pi} c^3 \rho_0 \cdot \frac{1}{\varepsilon_0}$ $\quad$ 1,0

$\Rightarrow \vec{E}_2 = \frac{c^3\rho_0}{3\varepsilon_0 r^2}\hat{r}$ //

1 Punto por cada $\vec{E}_i$
- $\to$ 0,5 flujo
- $\to$ 0,5 carga

$a < r < b$ | $\vec{E}_3 = 0$ ya que es conductor $\quad$ 1,0

$b < r < \infty$ | $\vec{E}_4 = \frac{c^3\rho_0}{3\varepsilon_0 r^2}\hat{r}$ //, equivalente a $\vec{E}_2$ $\quad$ 1,0

Luego, $U=\frac{\varepsilon_0}{2}\iiint_V |\vec{E}|^2 dv = \frac{\varepsilon_0}{2}\left[\iiint_{V|0\leq r<c} E_1^2 dv + \iiint_{V|c\leq r<a} E_2^2 dv + 0 + \iiint_{V|b\leq r<\infty} E_4^2 dv\right]$ 1,0 Por plantear la fórmula

$\therefore$

$$\iiint_{V|0\leq r<c} E_1^2 dv = \int_0^{2\pi}\int_0^{\pi}\int_0^{c}\left(\frac{r\rho_0}{3\varepsilon_0}\right)^2 r^2 \operatorname{sen}\theta \, dr\, d\theta\, d\phi = \frac{4\pi\rho_0^2}{9\varepsilon_0^2}\frac{c^5}{5}$$

0,3 por cálculos

$$\iiint_{V|c\leq r<a} E_2^2 dv = \int_0^{2\pi}\int_0^{\pi}\int_c^{a}\left(\frac{c^3\rho_0}{3\varepsilon_0 r^2}\right)^2 r^2 \operatorname{sen}\theta \, dr\, d\theta\, d\phi = \frac{4\pi c^6\rho_0^2}{9\varepsilon_0^2}\left(\frac{1}{c}-\frac{1}{a}\right)$$

0,3

$$\iiint_{V|b\leq r<\infty} E_4^2 dv = \int_0^{2\pi}\int_0^{\pi}\int_b^{\infty}\left(\frac{c^3\rho_0}{3\varepsilon_0 r^2}\right)^2 r^2 \operatorname{sen}\theta \, dr\, d\theta\, d\phi = \frac{4\pi c^6\rho_0^2}{9\varepsilon_0^2}\left(\frac{1}{b}-\cancel{\frac{1}{\infty}}^{\,0}\right) = \frac{4\pi c^6\rho_0^2}{9\varepsilon_0^2 b}$$

0,3

Finalmente: $U=\frac{\cancel{\varepsilon_0}}{\cancel{2}}\left[\frac{\cancel{4}^{2}\pi\rho_0^2}{9\varepsilon_0^{\cancel{2}}}\frac{c^5}{5} + \frac{\cancel{4}^{2}\pi c^6\rho_0^2}{9\varepsilon_0^{\cancel{2}}}\left(\frac{1}{c}-\frac{1}{a}\right) + \frac{\cancel{4}^{2}\pi c^6\rho_0^2}{9\varepsilon_0^{\cancel{2}} b}\right]$ 0,1 por concluir

$$U=\frac{2\pi\rho_0^2 c^5}{45\varepsilon_0} + \frac{2\pi c^6\rho_0^2}{9\varepsilon_0}\left(\frac{1}{c}-\frac{1}{a}\right) + \frac{2\pi c^6\rho_0^2}{9\varepsilon_0 b}$$