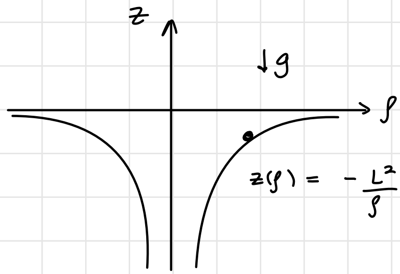
Pauta Aux 12

P1)



$\rho(t=0) = L$

$\vec{v}(t=0) = v_0\,\hat{\theta} = L\dot{\theta}_0\,\hat{\theta}$

Fuerzas : $\vec{N}$ → no tiene componente en $\hat{\theta}$ } ec. en $\hat{\theta}$) $m(\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta}) = 0$

$\vec{P}\text{eso} = -mg\,z(\rho)\,\hat{z}$

$$\Big\downarrow /\cdot\rho$$

$$m\rho^2\ddot{\theta} + 2\rho\dot{\rho}\dot{\theta} = 0$$

$$\underbrace{m\rho^2\dot{\theta} = m\rho_0^2\dot{\theta}_0 = mL^2}\frac{v_0}{L} \Longleftarrow \frac{d}{dt}\big(\underbrace{m\rho^2\dot{\theta}}_{\text{momento angular se conserva}}\big) = 0$$

$$\boxed{\dot{\theta} = \frac{Lv_0}{\rho^2}}$$

acá no tenemos ninguna fza no conservativa, ∴ sería conveniente trabajar con energía

$$E = \frac{1}{2}mv^2 + mg\,\underbrace{z(\rho)}_{\text{potencial gravitatoria}}$$

Para encontrar E necesito saber v^2 y resulta que la geometría del problema $z(\rho)$ y la cons. del momento angular me permite dejar todo en función de ρ

$$\vec{v} = \dot{\rho}\hat{\rho} + \rho\dot{\theta}\hat{\theta} + \dot{z}\hat{z} \overset{\text{lo voy a dejar todo en fn de }\rho}{=} \dot{\rho}\hat{\rho} + \rho\underbrace{\left(\frac{Lv_0}{\rho^2}\right)}_{\text{cons. de momento angular}}\hat{\theta} + \frac{d}{dt}\left(-\frac{L^2}{\rho}\right)\hat{z}$$

$$= \dot{\rho}\hat{\rho} + \frac{Lv_0}{\rho}\hat{\theta} + \frac{L^2}{\rho^2}\dot{\rho}\,\hat{z} \Longrightarrow v^2 = \dot{\rho}^2 + \frac{L^2v_0^2}{\rho^2} + \frac{L^4}{\rho^4}\dot{\rho}^2$$

∴

$$\boxed{E = \frac{1}{2}m\left(\dot{\rho}^2 + \frac{L^2v_0^2}{\rho^2} + \frac{L^4}{\rho^4}\dot{\rho}^2\right) + mg\left(-\frac{L^2}{\rho}\right)}$$

a) veamos la energía

$$E = \frac{1}{2} m \left(\dot{\rho}^2 + \frac{L^2 v_0^2}{\rho^2} + \frac{L^4}{\rho^4} \dot{\rho}^2 \right) + mg \left(-\frac{L^2}{\rho} \right)$$

Para que la partícula se mantenga a la misma altura $\dot{z} = 0 \Rightarrow \dot{\rho} = 0$

$\Rightarrow$

$$E = \underbrace{\frac{1}{2} m \left(\frac{L^2 v_0^2}{\rho^2} \right) + mg \left(-\frac{L^2}{\rho} \right)}_{\text{esto es lo que se le llama el "potencial efectivo"}}$$

y aquí viene el trucazo v_0 debe ser tal que $\rho = L$ es un punto de equilibrio

$$\frac{\partial E}{\partial \rho} = \frac{1}{2} m \frac{-2 L^2 v_0^2}{\rho^3} + mg \frac{L^2}{\rho^2}$$

$$\frac{\partial E}{\partial \rho}(\rho = L) = -m \frac{v_0^2}{L} + mg \overset{!}{=} 0 \quad \Rightarrow \quad v_0^2 = gL \quad , \quad v_0 = \sqrt{gL}$$

b)

$$\underset{\rho = L,\ \dot{\rho} = 0}{E(t=0)} = \frac{1}{2} m v_0^2 - mgL$$

y queremos saber cuál es la altura mínima, esto sucede cuando $\dot{z} = 0 \Rightarrow \dot{\rho} = 0$

$$E(\text{altura mínima}) = \frac{1}{2} m \frac{L^2 v_0^2}{\rho_{min}^2} - mg \frac{L^2}{\rho_{min}}$$

usamos cons. energía : $E(t=0) = E(\text{altura mín})$

$$\frac{1}{2} \cancel{m} v_0^2 - \cancel{m} gL = \frac{1}{2} \cancel{m} \frac{L^2 v_0^2}{\rho_{min}^2} - \cancel{m} g \frac{L^2}{\rho_{min}} \quad / \cdot \rho_{min}^2$$

$$\left(\frac{1}{2} v_0^2 - gL \right) \rho_{min}^2 = \frac{1}{2} L^2 v_0^2 - g L^2 \rho_{min}$$

ahora usamos $v_0 = \frac{1}{2}\sqrt{gL}$

$$\left(\frac{1}{8} \cancel{g} L - \cancel{g} L \right) \rho_{min}^2 = \frac{1}{2} L^2 \frac{\cancel{g} L}{4} - \cancel{g} L^2 \rho_{min}$$

$$0 = \frac{7}{8} L \rho_{min}^2 - L^2 \rho_{min} + \frac{L^3}{8}$$

$$0 = 7\rho_{min}^2 - 8L\rho_{min} + L^2$$

$$\rho_{min} = \frac{8L \pm \sqrt{64L^2 - 4\cdot 7\cdot L^2}}{2\cdot 7}$$

$$= L\left(\frac{4}{7} \pm \frac{1}{7}\sqrt{16-7}\right)$$

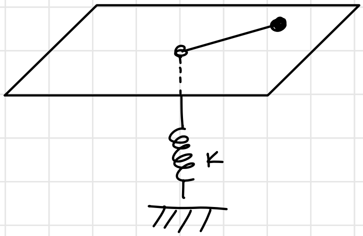
$$= \frac{L}{7}(4 \pm 3)$$

el mínimo es con el $-$

$$= \frac{L}{7}$$

$$\Longrightarrow z_{min} = -\frac{L^2}{\frac{L}{7}} = -7L \quad //$$

P2)



$\ell(t=0) = 0$ $\qquad \vec{v}(t=0) = v_0\,\hat{\theta} = \rho_0\dot{\theta}_0\,\hat{\theta}$

$\ell(\text{largo natural}) = 0$

a) ec. de mov.

$$m a_\rho = -K\rho$$

i) $\boxed{m(\ddot{\rho} - \rho\dot{\theta}^2) = -K\rho}$

$$m a_\theta = 0$$

ii) $\boxed{m(\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta}) = 0}$ $/\rho$

$$m\rho^2\ddot{\theta} + 2\rho\dot{\rho}\dot{\theta} = 0$$

$$\frac{d}{dt}\left(m\rho^2\dot{\theta}\right) = 0$$

$$m\rho^2\dot{\theta} = cte = m\rho_0^2\underbrace{\dot{\theta}_0}_{v_0/\rho_0}$$

$$m\rho^2\dot{\theta} = m\rho_0 v_0$$

$$\dot{\theta} = \frac{\rho_0 v_0}{\rho^2}$$

ii) $\Longrightarrow \quad m\left(\ddot{\rho} - \rho\left(\frac{\rho_0 v_0}{\rho^2}\right)^2\right) = -K\rho$

$$\boxed{\ddot{\rho} - \frac{\rho_0^2 v_0^2}{\rho^3} + \frac{K}{m}\rho = 0}$$

b)

en una órbita cirwnferencial $\rho = \rho_0 \Rightarrow \dot{\rho} = \ddot{\rho} = 0 \quad \Longrightarrow \quad -\frac{\rho_0^2 v_0^2}{\rho_0^3} + \frac{K}{m}\rho_0 = 0$

$$\frac{K}{m}\rho_0^2 = v_0^2$$

$$\rho_0 = \sqrt{\frac{m}{K}}\,v_0$$

$\epsilon(t) \ll 1$

c) Para esto, hacemos el cambio de variables $\rho(t) \to \rho_0 + \epsilon(t)$

$$\Longrightarrow \ddot{(\rho_0+\epsilon(t))} - \frac{\rho_0^2 v_0^2}{(\rho_0+\epsilon(t))^3} + \frac{K}{m}(\rho_0+\epsilon(t)) = 0$$

ρ_0 es cte $\qquad$ Taylor

$$\ddot{\epsilon}(t) - \left(\frac{\rho_0^2 v_0^2}{\rho_0^3} + \frac{-3\rho_0^2 v_0^2}{(\rho_0+\epsilon(t))^4}\bigg|_{\epsilon(t)=0} \cdot \epsilon(t) + \dots\right) + \frac{K}{m}(\rho_0+\epsilon(t)) = 0$$

$$\ddot{\epsilon} - \frac{v_0^2}{\rho_0} + \frac{3\rho_0^2 v_0^2}{\rho_0^4}\epsilon + \frac{K}{m}\rho_0 + \frac{K}{m}\epsilon$$

ahora usamos que $\rho_0 = \sqrt{\frac{m}{K}}\,v_0$

$$\ddot{\epsilon} - \frac{v_0^2}{\sqrt{\frac{m}{K}}\,v_0} + \frac{3v_0^2}{\frac{m}{K}v_0^2}\epsilon + \frac{K}{m}\sqrt{\frac{m}{K}}\,v_0 + \frac{K}{m}\epsilon = 0$$

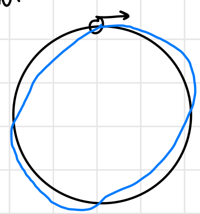
$$\ddot{\epsilon} - \cancel{\sqrt{\frac{k}{m}}\, v_0} + 3\frac{k}{m}\epsilon + \cancel{\sqrt{\frac{k}{m}}\, v_0} + \frac{k}{m}\epsilon = 0$$

$$\ddot{\epsilon} + \underbrace{\frac{4k}{m}}_{\omega_0^2}\epsilon = 0 \qquad \Longrightarrow \qquad \omega_0^2 = 4\frac{k}{m}$$

Recordar ec. oscilador armónico $\ddot{x} + \omega_0^2 x = 0$

d) Para que la órbita sea cerrada, el tiempo que se demora en dar una vuelta debe ser conmensurada con el tiempo que demora ρ en oscilar

visto de arriba



$$\overbrace{\frac{\text{Perímetro}}{v_0}}^{\text{periodo en dar una vuelta}} = n \overbrace{\frac{2\pi}{\omega_0}}^{T \text{ periodo de osc. de } \rho} \qquad \text{con} \quad n \in \mathbb{N}$$

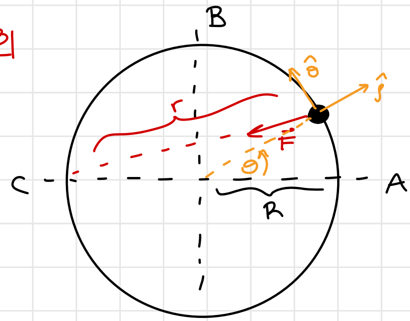
$$\frac{2\pi\rho_0}{v_0} = n\frac{2\pi}{2\sqrt{\frac{k}{m}}}$$

$$2\pi\frac{\cancel{\sqrt{\frac{m}{k}}}\,\cancel{v_0}}{\cancel{v_0}} = n\pi\cancel{\sqrt{\frac{m}{k}}}$$

$2\pi = n\pi$, vemos que se cumple para

$n = 0 \Rightarrow$ sí es cerrado

P3



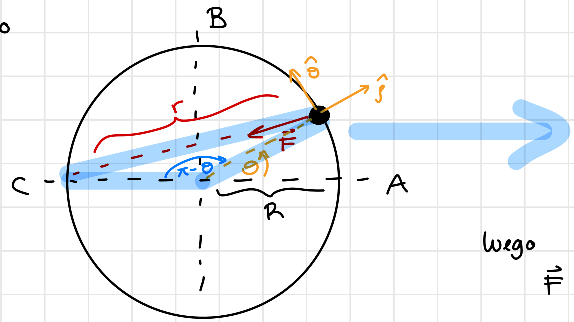
$F = \dfrac{k}{r^2}$, $W = \displaystyle\int_{camino} \vec{F}(\vec{r}) \cdot d\vec{r}$

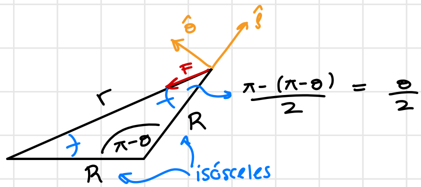
Para calcular el trabajo, tenemos que parametrizar el camino

$$\vec{r}(\theta) = R\,\hat{\rho} \quad , \theta \in [0, \tfrac{\pi}{2}]$$

$$d\vec{r} = R\,\hat{\theta}\,d\theta$$

Luego, queremos descomponer $\vec{F}$ en la base $\hat{\rho}, \hat{\theta}$ para poder hacer el producto punto





Luego

$$\vec{F} = F(r) \cdot \left(-\cos\tfrac{\theta}{2}\,\hat{\rho} + \sin\tfrac{\theta}{2}\,\hat{\theta}\right)$$

Y como nuestro único parámetro es θ, queremos escribir $F(r)$ en función de θ para esto, usando T^{ma} coseno :

$$r^2 = R^2 + R^2 - 2RR\,\underbrace{\cos(\pi-\theta)}_{-\cos\theta}$$

$$= 2R^2(1+\cos\theta)$$

Y ahora sí $\Longrightarrow F(r) = \dfrac{k}{2R^2(1+\cos\theta)}$

$$W = \int \vec{F}\cdot d\vec{r} = \int_0^{\pi/2} \underbrace{\frac{k}{2R^2(1+\cos\theta)}\left(-\cos\tfrac{\theta}{2}\,\hat{\rho} + \sin\tfrac{\theta}{2}\,\hat{\theta}\right)}_{\vec{F}} \cdot \underbrace{R\,d\theta\,\hat{\theta}}_{d\vec{r}}$$

$$= \int_0^{\pi/2} \frac{k}{2R^{\cancel{2}}(1+\cos\theta)} \sin\tfrac{\theta}{2}\,\cancel{R}\,d\theta$$

$$= \frac{k}{2R}\int_0^{\pi/2} \frac{\sin\theta/2}{1+\cos\theta}\,d\theta \qquad \cos^2\theta/2 = \frac{1+\cos\theta}{2}$$

$$= \frac{k}{2R}\int_0^{\pi/2} \frac{\sin\theta/2}{2\cos^2\theta/2}\,d\theta$$

$$= \frac{k}{2R} \cdot \left.\frac{1}{\cos\theta/2}\right|_0^{\pi/2} = \frac{k}{2R}\left(\underbrace{\frac{1}{\cos\pi/4}}_{\sqrt{2}} - \underbrace{\frac{1}{\cos 0}}_{1}\right) = \frac{k}{2R}(\sqrt{2}-1)$$