Aux 12 EDO

P1)

a) Escribimos el sistema de forma matricial

$$X' = \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix} X + \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

Primero solucionamos la EDO homogénea. Para ello buscamos los valores propios de $A = \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -3 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$-\left[(3-\lambda)(2+\lambda)\right] + 6 = 0$$

$$-\left[6 + \lambda - \lambda^2\right] + 6 = 0$$

$$\lambda^2 - \lambda = 0$$

$$\lambda(\lambda - 1) = 0$$

$\lambda_1 = 1$, $\lambda_2 = 0$. Con esto buscamos sus vectores propios asociados

v_1)
$$\begin{pmatrix} 3-\lambda_1 & -3 \\ 2 & -2-\lambda_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -3 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow 2x - 3y = 0$$
$$2x = 3y$$

$$v_1 = \begin{pmatrix} 1 \\ 2/3 \end{pmatrix}$$

$\underline{v_2}$

$$\begin{pmatrix} 3-0 & -3 \\ 2 & -2-0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow 3x = 3y \Rightarrow x = y$$

$$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Así $X = c_1 \begin{pmatrix} 1 \\ 2/3 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Con esto, la matriz fundamental es la siguiente:

$$\Phi(t) = \begin{pmatrix} e^t & 1 \\ \frac{2}{3}e^t & 1 \end{pmatrix}$$

Como necesitamos $\Phi^{-1}(t)$ para hacer variación de parámetros, la calculamos recordando que:

$$\Phi^{-1}(t) = \frac{1}{\det(\Phi(t))}\begin{pmatrix} 1 & -1 \\ -\frac{2}{3}e^t & e^t \end{pmatrix}$$

donde $\det(\Phi(t)) = e^t - \frac{2}{3}e^t = \frac{1}{3}e^t$

$$\Rightarrow \Phi^{-1}(t) = \begin{pmatrix} 3e^{-t} & -3e^{-t} \\ -2 & 3 \end{pmatrix}$$

Ahora usamos variación de parámetros para encontrar x_p

$$X_p = \Phi(t)\int \Phi^{-1}(t) F(t)\, dt$$

Primero vemos $\Phi^{-1}(t) F(t) = \begin{pmatrix} 3e^{-t} & -3e^{-t} \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ -1 \end{pmatrix}$

$$= \begin{pmatrix} 15e^{-t} \\ -11 \end{pmatrix}$$

Así $x_p = \Phi(t) \int \begin{pmatrix} 15e^{-t} \\ -11 \end{pmatrix} dt$

$$x_p = \begin{pmatrix} e^t & 1 \\ \frac{2}{3}e^t & 1 \end{pmatrix} \begin{pmatrix} -15e^{-t} \\ -11t \end{pmatrix} = \begin{pmatrix} -15 & -11t \\ -10 & -11t \end{pmatrix}$$

$$x_p = -5\begin{pmatrix} 3 \\ 2 \end{pmatrix} - 11t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Con lo y-e la sol general es

$$X = c_1\begin{pmatrix} 1 \\ 2/3 \end{pmatrix} e^t + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 11t\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 5\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

b) Valores propios de A

$$\begin{vmatrix} -\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = 0$$

$$-3\lambda + \lambda^2 + 2 = 0$$

$$(\lambda - 1)(\lambda - 2) = 0$$

$$\lambda_1 = 1 \quad , \quad \lambda_2 = 2$$

$\underline{V_1}$ $\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$-x + 2y = 0$$
$$x = 2y$$

$$V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\underline{V_2}$ $\begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow X_h = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

$$\Phi(t) = \begin{pmatrix} 2e^{t} & e^{2t} \\ e^{t} & e^{2t} \end{pmatrix}$$

$$\det(\Phi(t)) = 2e^{3t} - e^{3t} = e^{3t}$$

$$\Phi^{-1}(t) = e^{-3t}\begin{pmatrix} e^{2t} & -e^{2t} \\ -e^{t} & 2e^{t} \end{pmatrix} = \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{pmatrix}$$

$$x_p = \Phi(t)\int \Phi^{-1}(t)F(t)\,dt$$

$$\Phi^{-1}(t)F(t) = \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{pmatrix}\begin{pmatrix} e^{t} \\ -e^{t} \end{pmatrix} = \begin{pmatrix} 1+1 \\ -e^{-t}-2e^{-t} \end{pmatrix} = \begin{pmatrix} 2 \\ -3e^{-t} \end{pmatrix}$$

$$x_p = \Phi(t)\int\begin{pmatrix} 2 \\ -3e^{-t} \end{pmatrix}dt$$

$$= \begin{pmatrix} 2e^{t} & e^{2t} \\ e^{t} & e^{2t} \end{pmatrix}\begin{pmatrix} 2t \\ 3e^{-t} \end{pmatrix} = \begin{pmatrix} 4te^{t}+3e^{t} \\ 2te^{t}+3e^{t} \end{pmatrix} = 2te^{t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 3e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Sol general: $X = C_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t} + 2te^{t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 3e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

c) valores propios de A

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (1-\lambda)^2(3-\lambda) - (3-\lambda) = 0$$

$$= (3-\lambda)((1-\lambda)^2 - 1) = 0$$

$$= (3-\lambda)(1-2\lambda+\lambda^2-1) = 0$$

$$\lambda(\lambda-2)(3-\lambda) = 0$$

$\lambda_1 = 0, \quad \lambda_2 = 2, \quad \lambda_3 = 3$

Vectores propios:

$\underline{v_1}$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{matrix} x + y = 0 \\ 3z = 0 \end{matrix} \Rightarrow \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = v_1$$

$\underline{v_2}$

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{matrix} x - y = 0 \\ z = 0 \end{matrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$\underline{v_3}$

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{matrix} -2x + y = 0 \\ x - 2y = 0 \end{matrix}\right\} \Rightarrow x = y = 0$$

z libre $\qquad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Con esto $\quad x_h = c_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{3t}$

y $\quad \Phi(t) = \begin{pmatrix} 1 & e^{2t} & 0 \\ -1 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}$

Calculemos $\Phi^{-1}(t)$

$$\left(\begin{array}{ccc|ccc} 1 & e^{2t} & 0 & 1 & 0 & 0 \\ -1 & e^{2t} & 0 & 0 & 1 & 0 \\ 0 & 0 & e^{3t} & 0 & 0 & 1 \end{array}\right)$$

$$\Leftrightarrow \left(\begin{array}{ccc|ccc} 1 & e^{2t} & 0 & 1 & 0 & 0 \\ 0 & 2e^{2t} & 0 & 1 & 1 & 0 \\ 0 & 0 & e^{3t} & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 2e^{2t} & 0 & 1 & 1 & 0 \\ 0 & 0 & e^{3t} & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 1 & 0 & 1/2\, e^{-2t} & 1/2\, e^{-2t} & 0 \\ 0 & 0 & 1 & 0 & 0 & e^{-3t} \end{array}\right)$$

$$\Phi^{-1}(t) = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ e^{-2t} & e^{-2t} & 0 \\ 0 & 0 & 2e^{-3t} \end{pmatrix}$$

Calculemos x_p

$$x_p = \Phi(t) \int \Phi^{-1}(t) F(t)\, dt$$

$$\Phi^{-1}(t) F(t) = \begin{pmatrix} 1/2 & -1/2 & 0 \\ \frac{1}{2}e^{-2t} & \frac{1}{2}e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{pmatrix} \begin{pmatrix} e^{t} \\ e^{2t} \\ t e^{3t} \end{pmatrix}$$

$$\Phi^{-1}(t) F(t) = \begin{pmatrix} \frac{1}{2}e^{t} - \frac{1}{2}e^{2t} \\ \frac{1}{2}e^{-t} + 1/2 \\ t \end{pmatrix}$$

$$x_p = \Phi(t) \int \begin{pmatrix} 1/2\, e^{t} - 1/2\, e^{2t} \\ 1/2\, e^{-t} + 1/2 \\ t \end{pmatrix} dt$$

$$x_p = \begin{pmatrix} 1 & e^{2t} & 0 \\ -1 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 1/2\, e^{t} - 1/4\, e^{2t} \\ -1/2\, e^{-t} + 1/2\, t \\ t^2/2 \end{pmatrix}$$

$$X_p = \begin{pmatrix} 1/2\, e^{t} - 1/4\, e^{2t} - 1/2\, e^{t} + 1/2\, t e^{2t} \\ -1/2\, e^{t} + 1/4\, e^{2t} - 1/2\, e^{t} + 1/2\, t e^{2t} \\ \frac{t^2}{2} e^{3t} \end{pmatrix} = \begin{pmatrix} 1/2\, t e^{2t} - 1/4\, e^{2t} \\ \frac{1}{2} t e^{2t} + 1/4\, e^{2t} - e^{t} \\ \frac{t^2}{2} e^{3t} \end{pmatrix}$$

$$x_p = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} t e^{2t} + \begin{pmatrix} -1/4 \\ 1/4 \\ 0 \end{pmatrix} e^{2t} + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} e^{t} + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} t^2 e^{3t}$$

$$X = c_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{3t} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \frac{t}{2} e^{2t} + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \frac{e^{2t}}{4}$$

$$+ \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} e^{t} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \frac{t^2}{2} e^{3t}$$

P2) Tenemos que

$$W(t) = \begin{vmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{vmatrix} = x_{11} x_{22} - x_{12} x_{21}$$

Así $w'(t) = x_{11}' x_{22} + x_{11} x_{22}' - x_{12}' x_{21} - x_{12} x_{22}'$

$$w'(t) = x_{11}' x_{22} - x_{12} x_{21}' + x_{11} x_{22}' - x_{12}' x_{21}$$

$$w'(t) = \begin{vmatrix} x_{11}' & x_{21}' \\ x_{12} & x_{22} \end{vmatrix} + \begin{vmatrix} x_{11} & x_{21} \\ x_{12}' & x_{22}' \end{vmatrix}$$

$$w'(t) = \underbrace{\begin{vmatrix} a_{11} x_{11} + a_{12} x_{12} & a_{11} x_{21} + a_{12} x_{22} \\ x_{12} & x_{22} \end{vmatrix}}_{1} + \underbrace{\begin{vmatrix} x_{11} & x_{21} \\ a_{21} x_{11} + a_{22} x_{12} & a_{21} x_{21} + a_{22} x_{22} \end{vmatrix}}_{2}$$

Por ser $X_{1,2}$ sol de la EDO $X' = AX$

Luego, como sabemos de lineal, el det se mantiene haciendo operaciones entre columnas y filas.

Si en 1 tomamos $E_1' = E_1 - a_{12} E_2$ y en 2 $E_2' = E_2 - a_{21} E_1$ tenemos

$$w'(t) = \begin{vmatrix} a_{11} x_{11} & a_{11} x_{21} \\ x_{12} & x_{22} \end{vmatrix} + \begin{vmatrix} x_{11} & x_{21} \\ a_{22} x_{12} & a_{22} x_{22} \end{vmatrix}$$

$$w'(t) = a_{11} \begin{vmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{vmatrix} + a_{22} \begin{vmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{vmatrix}$$ por prop del det.

$$w'(t) = (a_{11} + a_{22}) W(t)$$

Quedándonos así una EDO en $W(t)$, cuya solución es

$$W(t) = W(t_0)\exp\left(\int_{t_0}^{t} (a_{11}(s)+a_{22}(s))\, ds\right)$$

con lo que concluimos

obs: esto se tiene por ser una EDO del tipo $x' = a(t)\, x$, cuya solución ya conocemos

Para el caso $n=3$ basta notar que también se cumple que

$$W'(t) = \begin{vmatrix} x_{11}' & x_{21}' & x_{31}' \\ x_{12} & x_{22} & x_{32} \\ x_{13} & x_{23} & x_{33} \end{vmatrix} + \begin{vmatrix} x_{11} & x_{21} & x_{31} \\ x_{12}' & x_{22}' & x_{32}' \\ x_{13} & x_{33} & x_{33} \end{vmatrix} + \begin{vmatrix} x_{11} & x_{21} & x_{31} \\ x_{12} & x_{22} & x_{32} \\ x_{13}' & x_{23}' & x_{33}' \end{vmatrix}$$