P3)

1) ¿Qué me piden?
- La energía electrostática

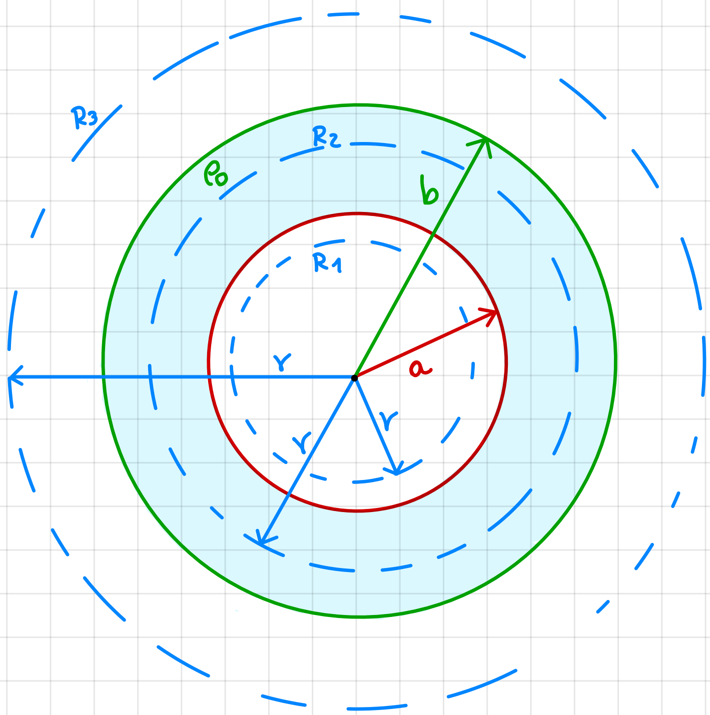
2) ¿Cómo se obtiene?
- $W = \frac{\varepsilon_0}{2} \int_{\text{todo el espacio}} |\vec{E}|^2 dv$

3) ¿Qué simetría tiene el problema?
- Esférica $\Rightarrow dv = r^2 \operatorname{sen}\theta \, dr \, d\theta \, d\phi$
$dA = r^2 \operatorname{sen}\theta \, d\theta \, d\phi$

4) ¿Cómo obtengo el campo?
- Al no pedirlo en un punto particular y al tener una simetría fácil, se hace usando la ley de Gauss:
$\oint_A \vec{E} \, d\vec{A} = \frac{1}{\varepsilon_0} \iiint_{\text{volumen}} \rho \, dv$

5) ¿Dónde integro?
$R_1: r \in [0, a)$
$R_2: r \in [a, b)$
$R_3: r \in [b, \infty)$

Diagrama del problema



Dado todo lo anterior empezamos a calcular el campo en cada región

Región R_1: $0 \leq r < a$

Usando la ley de Gauss: $\oint_A \vec{E} \, d\vec{A} = \cancel{\frac{q_{encerrada}}{\varepsilon_0}}^{\,0} \Rightarrow \boxed{\vec{E}(r) = 0 \, \hat{r} \quad ; \; 0 \leq r < a}$

la región no encierra carga

Región R_2: $a \leq r < b$

$$\oint_A \vec{E}\,d\vec{A} = \frac{1}{\varepsilon_0}\iiint_{\text{volumen}} \rho\,dv$$

$$\int_0^{2\pi}\int_0^{\pi} E(r)\, r^2 \operatorname{sen}\theta\, d\theta\, d\phi = \frac{1}{\varepsilon_0}\int_0^{2\pi}\int_0^{\pi}\int_a^{r} \rho_0\, r^2 \operatorname{sen}\theta\, dr\, d\theta\, d\phi$$

$$\cancel{4\pi}\, r^2 E(r) = \frac{\cancel{4\pi}}{\varepsilon_0}\rho_0 \int_a^{r} r^2\, dr$$

$$r^2 E(r) = \frac{\rho_0}{\varepsilon_0}\left(\frac{r^3}{3} - \frac{a^3}{3}\right)$$

$$\boxed{\vec{E}(r) = \frac{\rho_0}{\varepsilon_0}\left(\frac{r}{3} - \frac{a^3}{3r^2}\right)\hat{r} \quad ; \quad a \leq r < b}$$

Región R_3: $b \leq r < \infty$

$$\oint_A \vec{E}\,d\vec{A} = \frac{1}{\varepsilon_0}\iiint_{\text{volumen}} \rho\,dv$$

$$\int_0^{2\pi}\int_0^{\pi} E(r)\, r^2 \operatorname{sen}\theta\, d\theta\, d\phi = \frac{1}{\varepsilon_0}\int_0^{2\pi}\int_0^{\pi}\int_a^{b} \rho_0\, r^2 \operatorname{sen}\theta\, dr\, d\theta\, d\phi$$

$$\cancel{4\pi}\, r^2 E(r) = \frac{\cancel{4\pi}}{\varepsilon_0}\rho_0 \int_a^{b} r^2\, dr$$

$$r^2 E(r) = \frac{\rho_0}{\varepsilon_0}\left(\frac{b^3}{3} - \frac{a^3}{3}\right)$$

$$\boxed{\vec{E}(r) = \frac{\rho_0}{r^2\varepsilon_0}\left(\frac{b^3}{3} - \frac{a^3}{3}\right)\hat{r} \quad ; \quad b \leq r < \infty}$$

Resumen:

$$\vec{E}(r)=\begin{cases} 0\,\hat{r} & ;\ 0\leq r<a \\ \dfrac{\rho_0}{\varepsilon_0}\left(\dfrac{r}{3}-\dfrac{a^3}{3r^2}\right)\hat{r} & ;\ a\leq r<b \\ \dfrac{\rho_0}{r^2\varepsilon_0}\left(\dfrac{b^3}{3}-\dfrac{a^3}{3}\right)\hat{r} & ;\ b\leq r<\infty \end{cases}$$

Ahora que tenemos el campo podemos calcular la energía electrostática.

$$w=\frac{\varepsilon_0}{2}\int_{\substack{\text{todo el}\\ \text{espacio}}}|\vec{E}(r)|^2 dV$$

$$=\frac{\varepsilon_0}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}|\vec{E}(r)|^2 r^2 \operatorname{sen}\theta\, dr\, d\theta\, d\phi$$

$$=\frac{\varepsilon_0 4\pi}{2}\int_0^{\infty}|\vec{E}(r)|^2 r^2 dr$$

$$=2\pi\varepsilon_0\left(\int_0^a (E(r))^2 r^2 dr+\int_a^b (E(r))^2 r^2 dr+\int_b^\infty (E(r))^2 r^2 dr\right)$$

$$=2\pi\varepsilon_0\left(\cancelto{0}{\int_0^a (0)^2 r^2 dr}+\int_a^b\left(\frac{\rho_0}{\varepsilon_0}\left(\frac{r}{3}-\frac{a^3}{3r^2}\right)\right)^2 r^2 dr+\int_b^\infty\left(\frac{\rho_0}{r^2\varepsilon_0}\left(\frac{b^3}{3}-\frac{a^3}{3}\right)\right)^2 r^2 dr\right)$$

$$=2\pi\varepsilon_0\left(\frac{\rho_0^2}{\varepsilon_0^2}\int_a^b\left(\frac{r}{3}-\frac{a^3}{3r^2}\right)^2 r^2 dr+\frac{\rho_0^2}{\varepsilon_0^2}\left(\frac{b^3}{3}-\frac{a^3}{3}\right)^2\int_b^\infty\left(\frac{1}{r^2}\right)^2 r^2 dr\right)$$

$$=\frac{2\pi\rho_0^2}{\varepsilon_0}\left(\int_a^b\left(\frac{r}{3}-\frac{a^3}{3r^2}\right)^2 r^2 dr+\left(\frac{b^3}{3}-\frac{a^3}{3}\right)^2\int_b^\infty \frac{1}{r^4} r^2 dr\right)$$

$$= \frac{2\pi \rho_0^2}{\varepsilon_0} \left(\int_a^b \left(\frac{r^2}{9} - 2\frac{r}{3}\frac{a^3}{3r^2} + \frac{a^6}{9r^4} \right) r^2 dr + \left(\frac{b^3}{3} - \frac{a^3}{3} \right)^2 \int_b^\infty \frac{1}{r^2} dr \right)$$

$$= \frac{2\pi \rho_0^2}{9\varepsilon_0} \left(\int_a^b \left(r^2 - \frac{2a^3}{r} + \frac{a^6}{r^4} \right) r^2 dr + (b^3 - a^3)^2 \int_b^\infty \frac{1}{r^2} dr \right)$$

$$= \frac{2\pi \rho_0^2}{9\varepsilon_0} \left(\int_a^b \left(r^4 - 2a^3 r + \frac{a^6}{r^2} \right) dr + (b^3 - a^3)^2 \int_b^\infty \frac{1}{r^2} dr \right)$$

$$= \frac{2\pi \rho_0^2}{9\varepsilon_0} \left(\frac{r^5}{5}\Big|_a^b - 2a^3 \frac{r^2}{2}\Big|_a^b + a^6 \frac{1}{r}\Big|_b^a + (b^3 - a^3)^2 \frac{1}{r}\Big|_\infty^b \right.$$

$$= \frac{2\pi \rho_0^2}{9\varepsilon_0} \left(\frac{(b^5 - a^5)}{5} - a^3(b^2 - a^2) + a^6\left(\frac{1}{a} - \frac{1}{b} \right) + (b^3 - a^3)^2 \left(\frac{1}{b} - \cancelto{0}{\frac{1}{\infty}} \right) \right)$$

$$\boxed{W = \frac{2\pi \rho_0^2}{9\varepsilon_0} \left(\frac{(b^5 - a^5)}{5} - a^3(b^2 - a^2) + a^6\left(\frac{1}{a} - \frac{1}{b} \right) + (b^3 - a^3)^2 \frac{1}{b} \right)}$$