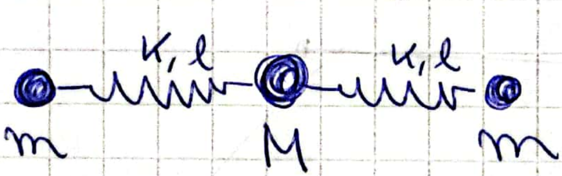
Pauta Pre Control 3

P1)

Molecula de CO_2



Queremos resolver este problema de masas acopladas por resortes

Significa que el mov de cada masa depende de las demás!

→ Pasos a seguir en estos problemas.

1) Escribir $F = ma$ para cada masa

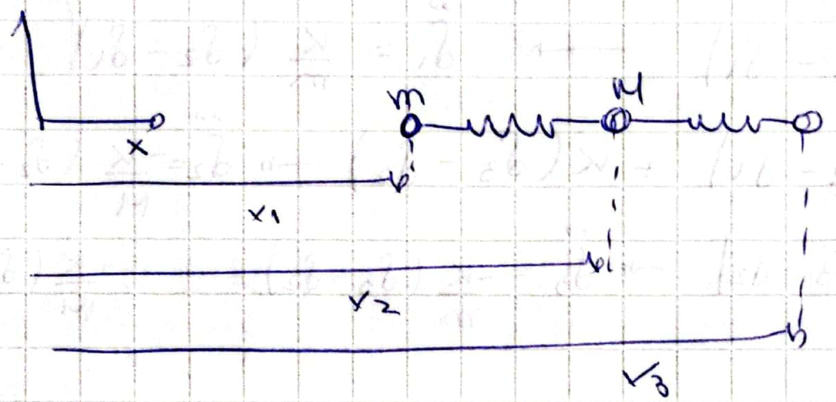
2) Escribirlo matricialmente → $\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \vdots \end{pmatrix} = \underbrace{\begin{pmatrix} & \end{pmatrix}}_{\text{matriz } A} \begin{pmatrix} x_1 \\ \vdots \end{pmatrix}$

(Algebra lineal)

3) Encontrar autovalores (serán las frecuencias de oscilación) y autovectores (modos normales)

$\det(A + \lambda I) = 0$ encontramos ωs

① Escribimos $F = m \cdot a$ para todas las masas:



$$\rightarrow m\ddot{x}_1 = +k(x_2 - x_1 - l)$$

$$M\ddot{x}_2 = -k(x_2 - x_1 - l) + k(x_3 - x_2 - l)$$

$$m\ddot{x}_3 = -k(x_3 - x_2 - l)$$

Ahora redefinimos x_1, x_2, x_3 para eliminar ctes y que el problema puede ser descrito tipo $\underline{\ddot{X} = AX}$

$$x_1 = \overset{\text{cte}}{\overline{x}_1} + \delta_1 \qquad \ddot{x}_1 = \ddot{\delta}_1$$

$$x_2 = \overline{x}_1 + l + \delta_2 \qquad \ddot{x}_2 = \ddot{\delta}_2$$

$$x_3 = \overline{x}_1 + 2l + \delta_3 \qquad \ddot{x}_3 = \ddot{\delta}_3$$

[illegible] ecuaciones de movimiento [illegible]

$$m\ddot{\delta}_1 = k(\delta_2 - \delta_1) \longrightarrow \ddot{\delta}_1 = \frac{k}{m}(\delta_2 - \delta_1)$$

$$M\ddot{\delta}_2 = -k(\delta_2 - \delta_1) + k(\delta_3 - \delta_2) \rightarrow \ddot{\delta}_2 = \frac{k}{M}\ [illegible]$$

$$m\ddot{\delta}_3 = -k(\delta_3 - \delta_2) \rightarrow \ddot{\delta}_3 = -\frac{k}{m}(\delta_3 - \delta_2)\ [illegible]$$

[illegible] Matriz [illegible]:

$$\begin{pmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \\ \ddot{\delta}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -k/m & k/m & 0 \\ k/M & -2k/M & k/M \\ 0 & k/m & -k/m \end{pmatrix}}_{A} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix}$$

Ahora para encontrar frecuencias de oscilación

$$\det(A + \underbrace{\lambda}_{\text{serán las frecuencias}} I) = 0$$

Con estos cambios:

$$m\ddot{\delta}_1 = K(\delta_2 - \delta_1) \longrightarrow \ddot{\delta}_1 = \frac{K}{m}(\delta_2 - \delta_1)$$

$$M\ddot{\delta}_2 = -K(\delta_2 - \delta_1) + K(\delta_3 - \delta_2) \longrightarrow \ddot{\delta}_2 = -\frac{K}{M}(\delta_2 - \delta_1) + \frac{K}{M}(\delta_3 - \delta_2)$$

$$m\ddot{\delta}_3 = -K(\delta_3 - \delta_2) \rightarrow \ddot{\delta}_3 = -\frac{K}{m}(\delta_3 - \delta_2)$$

② Matricialmente

$$\begin{pmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \\ \ddot{\delta}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -K/m & K/m & 0 \\ K/M & -2K/M & K/M \\ 0 & K/m & -K/m \end{pmatrix}}_{A} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix}$$

Ahora para encontrar frecuencias de oscilación:

$$\det(A + \lambda I) = 0$$

λ → serán las frecuencias.

$$\det \begin{pmatrix} -k/m + \lambda & k/m & 0 \\ k/M & -\frac{2k}{M} + \lambda & k/M \\ 0 & k/m & -k/m + \lambda \end{pmatrix} = 0$$

$$(-k/m + \lambda)\left[\left(-\frac{2k}{M} + \lambda\right)\left(-\frac{k}{m} + \lambda\right) - \frac{k}{M} \cdot \frac{k}{m}\right]$$

$$- \frac{k}{m} \cdot \left[\frac{k}{M}\left(-\frac{k}{m} + \lambda\right) - 0 \cdot \frac{k}{M}\right] = 0$$

llamemos $\omega_1^2 = k/m$, $\omega_2^2 = \frac{k}{M}$

$$(-\omega_1^2 + \lambda)\left[(-2\omega_2^2 + \lambda)(-\omega_1^2 + \lambda) - \omega_2^2\omega_1^2\right]$$

$$- \omega_1^2\left(\omega_2^2(-\omega_1^2 + \lambda)\right) = 0$$

factorizamos por $(-\omega_1^2 + \lambda)$

$$(-\omega_1^2 + \lambda)\left[(-2\omega_2^2 + \lambda)(-\omega_1^2 + \lambda) - \omega_2^2\omega_1^2 - \omega_1^2\omega_2^2\right] = 0$$

$$(-\omega_1^2 + \lambda)\left[(2\cancel{\omega_2^2\omega_1^2} - 2\omega_2^2\lambda - \omega_1^2\lambda + \lambda^2 - \cancel{2\omega_1^2\omega_2^2}\right] = 0$$

$$(-\omega_1^2+\lambda)\left(\lambda^2-\lambda(2\omega_2^2+\omega_1^2)\right)=0$$

$$(-\omega_1^2+\lambda)\left(\lambda(\lambda-2\omega_2^2-\omega_1^2)\right)=0$$

$$\rightarrow \quad \underbrace{\begin{array}{l} \lambda = 0 \\ \lambda = \omega_1^2 \quad - \quad k/m \\ \lambda = 2\omega_2^2+\omega_1^2 = \dfrac{2K}{M}+\dfrac{K}{m} \end{array}}_{\text{3 frecuencias de oscilación!}}$$

Ahora encontremos los modos normales

Si $\lambda = 0$

$$\rightarrow \begin{array}{c} ① \\ ② \\ ③ \end{array} \begin{pmatrix} -k/m & k/m & 0 \\ k/M & -2k/M & k/M \\ 0 & k/m & -k/m \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

de ① $\rightarrow -k/m\,\delta_1 + k/m\,\delta_2 = 0 \Rightarrow \delta_1 = \delta_2$

de ③ $\rightarrow k/m\,\delta_2 + -k/m\,\delta_3 = 0 \rightarrow \delta_2 = \delta_3$ $\quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

→ no hay oscilaciones solo se mueven juntos o estan quieta.

$$\longrightarrow \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} C$$

si $\lambda = K/m$

$$\rightarrow \begin{array}{c} ① \\ ② \\ ③ \end{array} \begin{pmatrix} 0 & K/m & 0 \\ K/M & \frac{2K}{M} - \frac{K}{m} & K/M \\ 0 & K/m & 0 \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

de ① o ③ $K/m\,\delta_2 = 0 \rightarrow \delta_2 = 0$

luego de ② $\rightarrow \underline{\delta_1 = -\delta_3}$

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-i\lambda t}$$

por ultimo si $\lambda = \frac{2K}{M} + \frac{K}{m}$

$$\begin{array}{c} ① \\ ② \\ ③ \end{array} \begin{pmatrix} -\frac{2K}{M} & K/m & 0 \\ \frac{K}{M} & -\frac{K}{m} & K/M \\ 0 & K/m & -\frac{2K}{M} \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

de ① $\quad -\frac{2K}{M}\delta_1 + \frac{K}{m}\delta_2 = 0$

$$\delta_1 = \frac{-M}{2m}\delta_2 \Rightarrow \delta_2 = \frac{-2m}{M}\delta_1$$

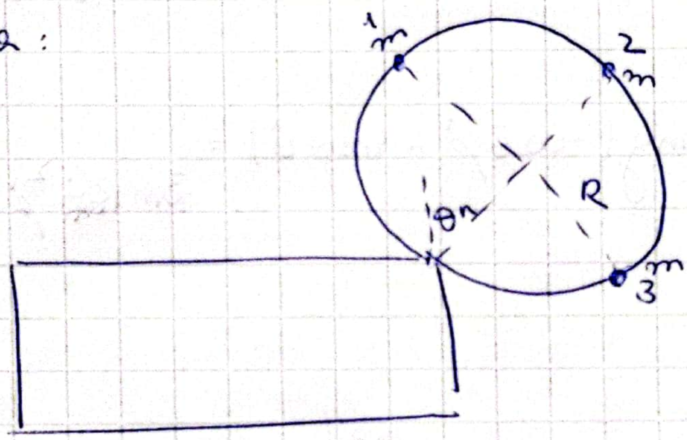
de ③ $\quad \frac{K}{m}\delta_2 - \frac{2K}{M}\delta_3 = 0$

$$\delta_3 = \frac{-M}{2m}\delta_2$$

$$\left(\begin{pmatrix} 1 \\ -2m/M \\ 1 \end{pmatrix} e^{-i\lambda t} \right) = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix}$$

P2)

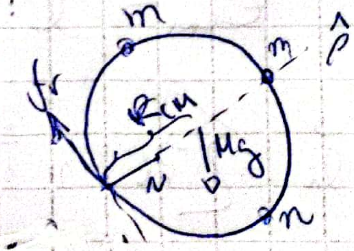
Tenemos:



Queremos calcular $\vec{fr}$ y $\vec{N}$ en función de θ_n

① Anotemos todas las fuerzas que existen:

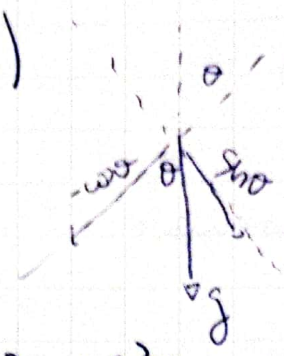
tomamos C.P:



$$\sum F = M\left[\left(\cancel{\ddot{R}}_{cm} - R_{cm}\dot{\theta}^2\right)\hat{\rho} + \left(2\cancel{\dot{R}}_{cm}\dot{\theta} + R_{cm}\ddot{\theta}\right)\hat{\theta}\right]$$

$$N\hat{\rho} - fr\hat{\theta} + M\vec{g} = - \quad + \left(2\cancel{\dot{R}}_{cm}\dot{\theta} + R_{cm}\ddot{\theta}\right)\hat{\theta}$$

donde $\vec{g} = g(-\cos\theta\,\hat{\rho} + \sin\theta\,\hat{\theta})$



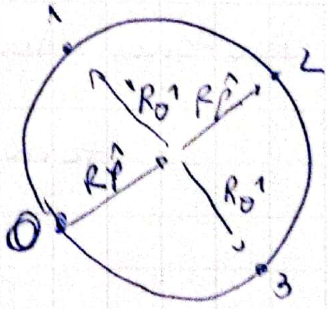
$\rightarrow N\hat{\rho} - fr\hat{\theta} + 3mg(-\cos\theta\,\hat{\rho} + \sin\theta\,\hat{\theta}) = -3mR_{cm}\dot{\theta}^2\hat{\rho} + 3mR_{cm}\ddot{\theta}\hat{\theta}$

separando con ejes:

$$\boxed{\begin{aligned} N - 3mg\cos\theta &= -MR_{cm}\dot{\theta}^2 \quad (1) \\ -fr + 3mg\sin\theta &= MR_{cm}\ddot{\theta} \quad (2) \end{aligned}}$$

Necesitamos otras ecuaciones, falta $\dot{\theta}^2$ y $\ddot{\theta}$

$$\vec{\tau}_{cm} = \dot{\vec{l}}$$



$$\vec{l} = \sum \vec{l}_i = \sum(\vec{r}_i \times m\vec{v}_i)$$

$$= (R\hat{\rho} - R\hat{\theta}) \times m(R\dot{\theta}(\hat{\rho} + \hat{\theta}))$$

$$+ 2R\hat{\rho} \times m2R\dot{\theta}\hat{\theta}$$

$$+ (R\hat{\rho} + R\hat{\theta}) \times m(R\dot{\theta}(-\hat{\rho} + \hat{\theta}))$$

$$\vec{l} = (mR^2\dot{\theta} + mR^2\dot{\theta} + 4mR^2\dot{\theta} + mR^2\dot{\theta} + mR^2\dot{\theta})\hat{k}$$

$$\vec{l} = 8mR^2\dot{\theta}\ \hat{k}$$

$$\boxed{\dot{\vec{l}} = 8mR^2\ddot{\theta}\ \hat{k}}$$

por otra parte el torque desde el punto O sera solo del peso $\longrightarrow$

$$\vec{\tau}_{cm} = \vec{R}_{cm} \times 3m\vec{g}$$

$$\vec{R}_{cm} = \sum \frac{m\vec{r}_i}{M} = \frac{1}{3m}\left(m(R\hat{\rho} - \cancel{R\hat{\theta}}) + m2R\hat{\rho} + m(R\hat{\rho} + \cancel{R\hat{\theta}})\right)$$

$$\boxed{\vec{R}_{cm} = \frac{4}{3}R\hat{\rho}}$$

$$\rightarrow \vec{\tau}_{cm} = \frac{4}{3}R\hat{\rho} \times (3mg(-\cos\theta\hat{\rho} + \sin\theta\ \hat{\theta}))$$

$\boxed{\vec{\tau}_{cm} = 4Rmg\sin\theta\ \hat{k}}$ igualando tenemos:

Escaneado con CamScanner

$$\boxed{8mR^2\ddot{\theta} = 4mgR\,\text{sen}\,\theta}$$ otra ec

$$\rightarrow \boxed{\ddot{\theta} = \frac{g}{2R}\,\text{sen}\,\theta}$$

Falta $\dot{\theta}^2$

$$\longrightarrow \quad 8mR^2\,\frac{d\dot{\theta}}{d\theta}\,\dot{\theta} = 4mgR\,\text{sen}\,\theta$$

$$\frac{d\dot{\theta}}{\cancel{d\theta}}\,\dot{\theta} = \frac{g}{2R}\,\text{sen}\,\theta\,d\theta$$

$$\frac{\dot{\theta}^2}{2} = \frac{-g}{2R}\cos\theta$$

$$\boxed{\dot{\theta}^2 = -g/R\,\cos\theta}$$

Reemplazando en ① y ②

$$N - 3mg\cos\theta = + \cancel{M}\underbrace{\dot{\theta}^2 \cdot R_{cm}}$$

$$\frac{\cancel{3}mt\,g}{R}\cos\theta \cdot \frac{4R}{\cancel{3}}$$

$$\boxed{N = 7mg\cos\theta}$$

$$-f_R + 3mg\,\text{sen}\,\theta = MR_{cm}\ddot{\theta}$$

$$= \cancel{3}m \cdot \frac{\overset{2}{\cancel{4}}}{\cancel{3}}\cancel{R}\,\frac{\cancel{g}}{\cancel{2R}}\,\text{sen}\,\theta$$

$$-f_R + 3mg\,\text{sen}\,\theta = 2mg\,\text{sen}\,\theta$$

$$\boxed{f_r = mg\,\text{sen}\,\theta}$$

Escaneado con CamScanner