Pauta Auxiliar 15

P1 (a) $\hat{u}(x,y) = u(r,\theta)$, como $x = r\cos\theta$
$y = r\sin\theta$

$\Rightarrow \hat{u}(x,y) = \hat{u}(r\cos\theta, r\sin\theta) = u(r,\theta)$ ✓

$$\frac{\partial \hat{u}}{\partial x} = \frac{\partial \hat{v}}{\partial y} \quad y \quad \frac{\partial \hat{u}}{\partial y} = -\frac{\partial \hat{v}}{\partial x} \quad (C-R)$$

$$\frac{\partial u}{\partial r} = \frac{\partial \hat{u}}{\partial r} = \frac{\partial \hat{u}}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial \hat{u}}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial \hat{u}}{\partial x}\cdot\cos\theta + \frac{\partial \hat{u}}{\partial y}\cdot\sin\theta$$

$$\frac{\partial u}{\partial \theta} = \dots = \frac{\partial \hat{u}}{\partial x}\cdot(-r\sin\theta) + \frac{\partial \hat{u}}{\partial y}\cdot(r\cos\theta)$$

Similar con $\frac{\partial v}{\partial r}$ y $\frac{\partial v}{\partial \theta}$

$\Rightarrow$ Si $\frac{\partial \hat{u}}{\partial x} = \frac{\partial \hat{v}}{\partial y}$ y $\frac{\partial \hat{u}}{\partial y} = -\frac{\partial \hat{v}}{\partial x}$

$$\Rightarrow \frac{\partial v}{\partial r} = \frac{\partial \hat{v}}{\partial x}\cdot\cos\theta + \frac{\partial \hat{v}}{\partial y}\cdot\sin\theta = \frac{\partial \hat{u}}{\partial x}\cdot\sin\theta - \frac{\partial \hat{u}}{\partial y}\cdot\cos\theta$$

$$\Rightarrow -\frac{\partial v}{\partial r} = \frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial \hat{v}}{\partial x}\cdot(-r\sin\theta) + \frac{\partial \hat{v}}{\partial y}(r\cos\theta)$$

$$= \frac{\partial \hat{u}}{\partial x}\cdot(r\cos\theta) + \frac{\partial \hat{u}}{\partial y}(r\sin\theta)$$

$$\Rightarrow \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial r}$$

$$\Leftarrow \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \iff \frac{\partial \hat{u}}{\partial x}\cdot\cos\theta + \frac{\partial \hat{u}}{\partial y}\cdot\sin\theta = -\frac{\partial \hat{v}}{\partial x}\cdot\sin\theta + \frac{\partial \hat{v}}{\partial y}\cos\theta$$

$$\iff (1)\ \left(\frac{\partial \hat{u}}{\partial x} - \frac{\partial \hat{v}}{\partial y}\right)\cos\theta = -\left(\frac{\partial \hat{u}}{\partial y} + \frac{\partial \hat{v}}{\partial x}\right)\sin\theta$$

$$-\frac{\partial v}{\partial r} = \frac{1}{r}\frac{\partial u}{\partial \theta} \iff -\frac{\partial \hat{v}}{\partial x}\cdot\cos\theta - \frac{\partial \hat{v}}{\partial y}\sin\theta = -\frac{\partial \hat{u}}{\partial x}\cdot\sin\theta + \frac{\partial \hat{u}}{\partial y}\cdot\cos\theta$$

$$\iff (2)\ \left(\frac{\partial \hat{u}}{\partial x} - \frac{\partial \hat{v}}{\partial y}\right)\sin\theta = \left(\frac{\partial \hat{u}}{\partial y} + \frac{\partial \hat{v}}{\partial x}\right)\cos\theta$$

$$(1)^2 + (2)^2 \Rightarrow \quad \left(\frac{\partial \hat{u}}{\partial x} - \frac{\partial \hat{v}}{\partial y}\right)^2 = \left(\frac{\partial \hat{u}}{\partial y} + \frac{\partial \hat{v}}{\partial x}\right)^2$$

$$\Rightarrow \boxed{\frac{\partial \hat{u}}{\partial x} = \frac{\partial \hat{v}}{\partial y} \quad y \quad \frac{\partial u}{\partial y} = -\frac{\partial \hat{v}}{\partial x}} \ \text{C-R}$$

$z = re^{i\theta}$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial r} \iff \frac{1}{r} = \frac{\partial f}{\partial z} \cdot e^{i\theta}$$

$$\Rightarrow \frac{1}{re^{i\theta}} = f'(z) \iff \boxed{\frac{1}{z} = f'(z)}$$

P2 a) $\sum_{n=1}^{\infty} \frac{n^n}{n!} z^n \Rightarrow R = \limsup_{n\to\infty} \frac{\frac{n^n}{\cancel{n!}}}{\frac{(n+1)^{n+1}}{\cancel{(n+1)!}}}$

$$\Rightarrow R = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \longrightarrow \frac{1}{e}$$

$$\Rightarrow \boxed{R = \frac{1}{e}}$$

b)

$$|a_n| = 1 \rightarrow \boxed{R = \limsup_{n\to\infty} 1 = 1}$$

c)

$$2 \le \sqrt[n]{n+2^n} \le \sqrt[n]{2^n+2^n} = \sqrt[n]{2} \cdot 2 \longrightarrow 2$$

$$\Rightarrow \boxed{R = \frac{1}{2}}$$

d) ~~Creciente~~ $|a_n| = \log(n)^2$

$1 \leq |a_n| \leq n^2$

$\Rightarrow \sqrt[n]{1} = 1 \leq \sqrt[n]{|a_n|} \leq \sqrt[n]{n^2} \Longrightarrow \sqrt[n]{a_n} \rightarrow 1$

$\Longrightarrow \boxed{R = 1}$

e) Pendiente

f) $a_n = (3+(-1)^n)^n = \begin{cases} 4^n \\ 2^n \end{cases}$

$\rightarrow \sqrt[n]{|a_n|} = \begin{cases} 4 & \text{si } n \text{ par} \\ 2 & n \text{ impar} \end{cases}$

$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 4 \longrightarrow \boxed{R = \frac{1}{4}}$

P3

1) $\dfrac{1}{1-z} = \sum_{k=0}^{\infty} z^k$ ✓ Centrado en $z_0=0$

Tiene $\boxed{R=1}$ $\boxed{z_0=0}$

2) $\dfrac{1}{1-z} = \dfrac{1}{1-(z+1)+1} = \dfrac{1}{2-(z+1)} = \dfrac{1}{2}\cdot\dfrac{1}{1-\left(\frac{z+1}{2}\right)}$

$= \dfrac{1}{2}\sum_{k=0}^{\infty}\left(\dfrac{z+1}{2}\right)^k = \sum_{k=0}^{\infty}\left(\dfrac{1}{2}\right)^{k+1}(z+1)^k$

$R = \limsup_{n\to\infty} \dfrac{\left(\cancel{\frac{1}{2}}\right)^{k+1}}{\left(\cancel{\frac{1}{2}}\right)^{k+2}} = \boxed{2}$

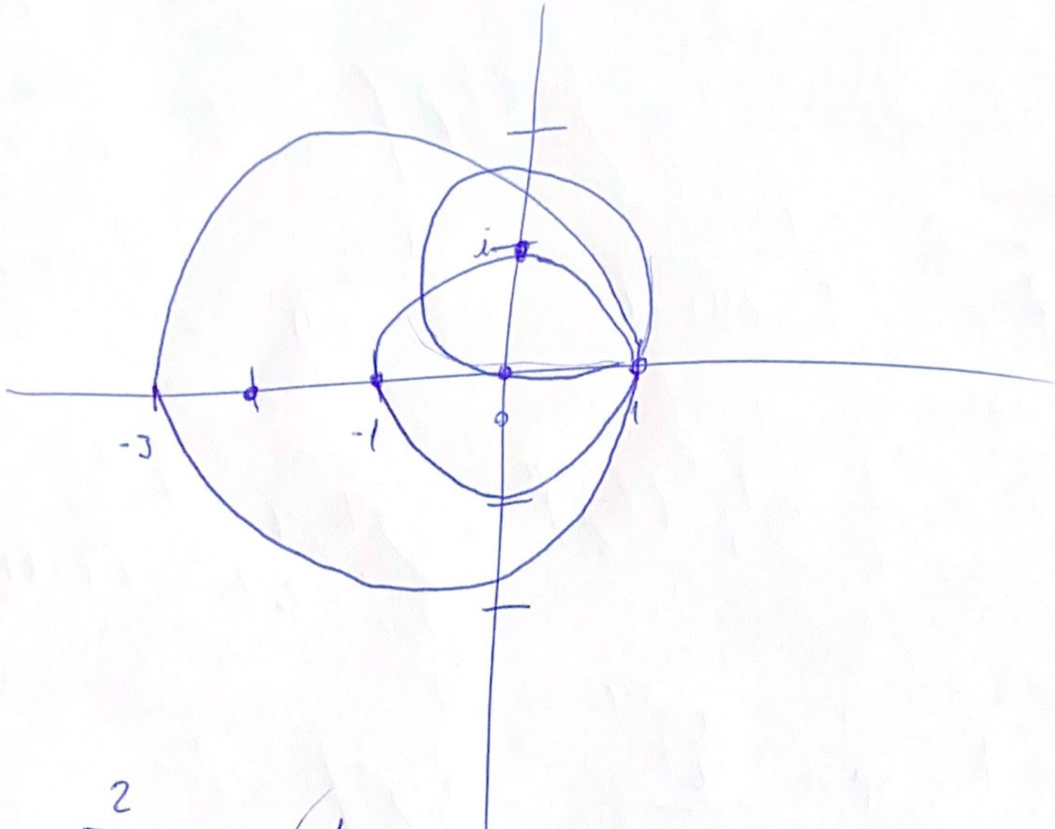
$\boxed{z_0=-1}$

3) $\dfrac{1}{1-z} = \dfrac{1}{1-(z+i)-i} = \dfrac{1}{1-i-(z-i)} = \dfrac{1}{1-i}\cdot\dfrac{1}{1-\left(\frac{z-i}{1-i}\right)}$

$= \dfrac{1}{1-i}\sum_{k=0}^{\infty}\left(\dfrac{z-i}{1-i}\right)^k = \sum_{k=0}^{1}\left(\dfrac{1}{1-i}\right)^{k+1}(z-i)^k$

$R = \limsup \dfrac{\left(\left|\frac{1}{1-i}\right|\right)^{k+1}}{\left(\left|\frac{1}{1-i}\right|\right)^{k+2}} = \sqrt{2}$ y $\boxed{z_0=i}$

Grafico idea Todos los circulos limitados por $\boxed{z=1}$



$\boxed{P4}$

$$\frac{2}{z^2+1} = \frac{1}{-i}\left(\frac{1}{z+i} - \frac{1}{z-i}\right) = i\left(\frac{1}{z+i} - \frac{1}{z-i}\right)$$

$$= i\left(\frac{-1}{-1-i-(z-1)} + \frac{1}{-1+i-(z-1)}\right)$$

$$= i\left[\left(\frac{1}{1-\frac{(z-1)}{-(1+i)}}\right)\cdot\frac{1}{(1+i)} + \left(\frac{1}{1-\frac{(z-1)}{(i-1)}}\right)\frac{1}{i-1}\right]$$

$$= i\left[\frac{1}{1+i}\sum_{k=0}^{\infty} \frac{(z-1)^k}{(-1)^k(1+i)^k} + \frac{1}{i-1}\sum_{k=0}^{\infty}\frac{(z-1)^k}{(i-1)^k}\right]$$

$$= i \sum_{k=0}^{\infty} \left[\frac{(-1)^k}{(1+i)^{k+1}} + \frac{1}{(i-1)^{k+1}} \right] (z-1)^k$$