Pauta Auxiliar 11

P1 a) $\widehat{f(x)}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\xi x}\cdot \mathbb{1}_{[-1,1]}(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-i\xi x}\,dx$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-1}^{1}\cos(\xi x) + i\,\cancel{\sin(\xi x)}^{\,0}\,dx\right]$$

$$= \frac{2}{\sqrt{2\pi}}\left.\frac{\sin(\xi x)}{\xi}\right|_{0}^{1} = \boxed{\frac{2\sin(\xi)}{\sqrt{2\pi}\;\xi}}$$

b) Idea

$$\widehat{g(x)}(\xi) = \widehat{f(x)}(\xi)\cdot e^{-\xi^2} = \widehat{f(x)}(\xi)\cdot\widehat{h(x)}(\xi) = \widehat{\left(\frac{f*h}{\sqrt{2\pi}}\right)}(\xi)$$

Buscar $\widehat{h(x)}(\xi) = e^{-\xi^2} = \frac{\sqrt{\frac{1}{2}}}{\sqrt{\frac{1}{2}}}\cdot e^{-\xi^2} = \frac{\sqrt{\frac{1}{2}}}{\sqrt{2\cdot\frac{1}{4}}}\cdot e^{-\frac{\xi^2}{4\cdot\frac{1}{4}}}$

$$= \sqrt{\frac{1}{2}}\cdot\widehat{e^{-\frac{x^2}{4}}}(\xi)$$

Anti Transformada

$\Rightarrow$ $\boxed{h(x) = \sqrt{\frac{1}{2}}\,e^{-\frac{x^2}{4}}}$

$$\Rightarrow g(x) = \frac{f*h}{\sqrt{2\pi}} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\mathbb{1}_{[-1,1]}(y)\cdot\frac{1}{\sqrt{2}}\cdot e^{-\frac{(x-y)^2}{4}}\,dy = \boxed{\frac{1}{2\sqrt{\pi}}\cdot\int_{-1}^{1}e^{-\frac{(x-y)^2}{4}}\,dy}$$

P2 a)

$$\widehat{e^{-ax}\mathbb{1}_{\{x>0\}}}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax}\,\mathbb{1}_{[0,\infty)}(x)\cdot e^{-i\xi x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-x(a+i\xi)}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left.\frac{e^{-x(a+i\xi)}}{-(a+i\xi)}\right|_0^{\infty} = \boxed{\frac{1/\sqrt{2\pi}}{a+i\xi}}$$

Propuesto usando propiedades como en el aux

b) $g(x) = g_1(x) + g_2(x)$

Con $g_1(x) = e^{-2x}\cdot\mathbb{1}_{\{x>0\}}$ y $g_2(x) = e^{5x}\mathbb{1}_{\{x<0\}}$

$\widehat{g_1(x)} = \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{2+i\xi}$ usando a)

Paso b) 2. Alternativas 1) Por definición (Propuesto)

2) Con propiedades. Notar que $g_2(-x) = e^{-5x}\cdot\mathbb{1}_{\{x>0\}}$

Usando a)

$$\Rightarrow \widehat{g_2(-x)}(\xi) = \frac{1}{\sqrt{2\pi}}\,\frac{1}{5+i\xi} \Rightarrow \widehat{g_2(x)}(\xi) = \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{5-i\xi}$$

$$\Rightarrow \boxed{\widehat{g(x)}(\xi) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{2+i\xi} + \frac{1}{5-i\xi}\right)}$$

P3 Suponer solución de la forma $u(x,t) = X(x) \cdot T(t)$

$$\Rightarrow \frac{T''(t) + a T(t)}{T(t)} = \frac{X''(x)}{X(x)} = c$$

$$\Rightarrow X''(x) - c X(x) = 0$$

$$X(0) = X(\pi) = 0$$

$\boxed{\text{Descartar que } c \geq 0}$

Como $c < 0 \Rightarrow c = -\tilde{c}$

$$\Rightarrow X''(x) + \tilde{c} X(x) = 0$$

$$\Rightarrow X(x) = A \cos(\sqrt{\tilde{c}}\, x) + B \sin(\sqrt{\tilde{c}}\, x)$$

$X(0) = 0 \Rightarrow A = 0$ y $X(\pi) = 0 \Rightarrow 0 = B \sin(\sqrt{\tilde{c}}\, \pi)$

$$\Rightarrow \sqrt{\tilde{c}}\, \pi = n \cdot \pi \Rightarrow \sqrt{\tilde{c}} = n \Rightarrow \boxed{\tilde{c} = n^2}$$

$$\Rightarrow \boxed{X_n(x) = B_n \cdot \sin(n x)}$$

$$T_n''(t) + a T_n'(t) + n^2 T_n(t) = 0$$

$$\Rightarrow T_n(t) = \left[D_n \cdot \cos\left(\frac{\sqrt{4n^2 - a^2}}{2} t \right) + E_n \sin\left(\frac{\sqrt{4n^2 - a^2}}{2} t \right) \right] e^{-\frac{a}{2} t}$$

$$\Rightarrow U_N(x,t) = e^{-\frac{a t}{2}}\left[\tilde{D}_N \cos\left(\frac{\sqrt{4N^2-a^2}}{2}t\right) + \tilde{E}_N \sin\left(\frac{\sqrt{4N^2-a^2}}{2}t\right)\right]\sin(Nx)$$

Por Superposición

$$\Rightarrow u(x,t) = \sum_{N=1}^{\infty} U_N(x,t)$$

Usando que $u(x,0) = \sin(x) + 5\sin(4x) = \sum_{N=1}^{\infty} a_N \sin(Nx)$ $\begin{cases} a_1 = 1 \\ a_4 = 5 \\ a_N = 0 \end{cases}$

$$\Rightarrow \cancel{e^{-\frac{a\cdot 0}{2}}}^{\,1}\,\tilde{D}_1 = 1 \Rightarrow \boxed{\tilde{D}_1 = 1}$$

$$e^{-\frac{a\cdot 0}{2}}\cdot \tilde{D}_4 = 5 \Rightarrow \boxed{\tilde{D}_4 = 5} \qquad \tilde{D}_j = 0 \quad \forall j \in \mathbb{N} - \{1,4\}$$

Por otro lado $u_t(x,t) = \sum_{N=1}^{\infty} e^{-\frac{a t}{2}}\left[\left(\left(\frac{-a}{2}\right)\tilde{D}_N + \frac{\sqrt{4N^2-a^2}}{2}\tilde{E}_N\right)\cos\left(\frac{\sqrt{4N^2-a^2}}{2}t\right) + \left(\left(\frac{-a}{2}\right)\tilde{E}_N - \frac{\sqrt{4N^2-a^2}}{2}\cdot\tilde{D}_N\right)\sin\left(\frac{\sqrt{4N^2-a^2}}{2}t\right)\right]\cdot\sin(Nx)$

$$\Rightarrow u_t(x,0) = \sum_{N=1}^{\infty}\left[\left(\frac{-a}{2}\right)\tilde{D}_N + \frac{\sqrt{4N^2-a^2}}{2}\tilde{E}_N\right]\sin(Nx) = 0 = \sum_{N=1}^{\infty} 0\cdot\sin(Nx)$$

$$\Rightarrow \left(\left(\frac{-a}{2}\right)\tilde{D}_N + \frac{\sqrt{4N^2-a^2}}{2}\tilde{E}_N\right) = 0 \Rightarrow \boxed{\tilde{E}_N = 0 \quad \forall N \in \mathbb{N} - \{1,4\}}$$

$$\text{y} \quad \left(\frac{-a}{2}\right) + \frac{\sqrt{4-a^2}}{2}\cdot\tilde{E}_1 = 0 \Rightarrow \boxed{\tilde{E}_1 = \frac{a}{\sqrt{4-a^2}}}$$

$$\left(\frac{-a}{2}\right)\cdot\tilde{D}_4 + \frac{\sqrt{4^3-a^2}}{2}\cdot\tilde{E}_4 = 0 \Rightarrow \boxed{\tilde{E}_4 = \frac{a}{\sqrt{4^3-a^2}}}$$

Con lo que se obtiene $\boxed{u(x,t)}$