P1 a) $f(x,y,z) = \dfrac{z}{x^2+y^2+z^2}$

$$\frac{\partial f}{\partial x} = z(x^2+y^2+z^2)^{-2}(-1)(2x) = \frac{-2xz}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial f}{\partial y} = z(x^2+y^2+z^2)^{-2}(-1)(2y) = -\frac{2yz}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial f}{\partial z} = (x^2+y^2+z^2)^{-1} + z(x^2+y^2+z^2)^{-2}(-1)(2z) = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}$$

$$\Rightarrow \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

Primero calcular la formula en esfericas

$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\hat{u} + \frac{1}{h_v}\frac{\partial f}{\partial v}\hat{v} + \frac{1}{h_w}\frac{\partial f}{\partial w}\hat{w}$$ / $h_u = 1$, $h_v = r\sin\phi$ y $h_w = r$

$\hat{u} = \hat{r}$, $\hat{v} = \hat{\theta}$ y $\hat{w} = \hat{\phi}$

g esferica

$$\Rightarrow \nabla g(r,\theta,\phi) = \frac{\partial g}{\partial r}\hat{r} + \left(\frac{\partial g}{\partial \theta}\right)\cdot\frac{1}{r\sin\phi}\hat{\theta} + \left(\frac{\partial g}{\partial \phi}\right)\cdot\frac{1}{r}\hat{\phi}$$

En nuestro caso $f(x,y,z) = g(r,\theta,\phi) = \dfrac{r\cos(\phi)}{r^4} = \dfrac{\cos(\phi)}{r^3}$

$$\Rightarrow \nabla g(r,\theta,\phi) = \frac{(-3)\cos(\phi)}{r^4}\hat{r} + \frac{-\sin(\phi)}{r^4}\hat{\phi} = \frac{-1}{r^4}\left(3\cos(\phi)\hat{r} + \sin(\phi)\hat{\phi}\right)$$

Se relacionan combinando $r\cos\theta\sin\phi = x$, $r\sin\theta\sin\phi = y$, $r\cos\phi = z$

$\Rightarrow$ $\hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$, etc

$\boxed{b}$ $r(\xi,\eta,\phi) = \left(\xi\eta\cos\phi,\ \xi\eta\sin\phi,\ \frac{1}{2}(\eta^2-\xi^2)\right)$

$$\frac{\partial r}{\partial \xi} = (\eta\cos\phi,\ \eta\sin\phi,\ -\xi) \Rightarrow h_\xi = \sqrt{\eta^2+\xi^2} \Rightarrow \underbrace{\left(\frac{\eta}{\sqrt{\eta^2+\xi^2}}\cos\phi,\ \frac{\eta}{\sqrt{\eta^2+\xi^2}}\sin\phi,\ \frac{-\xi}{\sqrt{\eta^2+\xi^2}}\right)}_{\hat{\xi}}$$

$$\frac{\partial r}{\partial \eta} = (\xi\cos\phi,\ \xi\sin\phi,\ \eta) \Rightarrow h_\eta = \sqrt{\eta^2+\xi^2} \Rightarrow \underbrace{\left(\frac{\xi}{\sqrt{\eta^2+\xi^2}}\cos\phi,\ \frac{\xi}{\sqrt{\eta^2+\xi^2}}\sin\phi,\ \frac{\eta}{\sqrt{\eta^2+\xi^2}}\right)}_{\hat{\eta}}$$

$$\frac{\partial r}{\partial \phi} = (-\xi\eta\sin\phi,\ \xi\eta\cos\phi,\ 0) \Rightarrow h_\phi = \xi\eta \Rightarrow \underbrace{(-\sin\phi,\ \cos\phi,\ 0)}_{\hat{\phi}}$$

$$\Rightarrow \boxed{\nabla g(\xi,\eta,\phi) = \frac{1}{\sqrt{\eta^2+\xi^2}}\left(\frac{\partial g}{\partial \xi}\hat{\xi} + \frac{\partial g}{\partial \eta}\hat{\eta}\right) + \frac{1}{\xi\eta}\frac{\partial g}{\partial \phi}\hat{\phi}}$$

$$\boxed{\Delta g(\xi,\eta,\phi) = \left(\frac{1}{\xi^2+\eta^2}\right)\left[\frac{1}{\xi}\frac{\partial g}{\partial \xi} + \frac{\partial^2 g}{\partial \xi^2} + \frac{1}{\eta}\frac{\partial g}{\partial \eta} + \frac{\partial^2 g}{\partial \eta^2}\right] + \frac{1}{(\xi\eta)^2}\frac{\partial^2 g}{\partial \phi^2}}$$

$\boxed{P2}$ $\operatorname{div}(f\vec{F}) = f \operatorname{div}\vec{F} + \vec{F}\cdot\nabla f$

Probar para cartesianas

$$\operatorname{div}(f\vec{F}) = \frac{\partial}{\partial x}(fF_1) + \frac{\partial}{\partial y}(fF_2) + \frac{\partial}{\partial z}(fF_3)$$

$$= \frac{\partial f}{\partial x}\cdot F_1 + f\frac{\partial F_1}{\partial x} + \frac{\partial f}{\partial y}\cdot F_2 + f\frac{\partial F_2}{\partial y} + \frac{\partial f}{\partial z}F_3 + f\frac{\partial F_3}{\partial z}$$

$$= f\cdot \operatorname{div}(f) + \vec{F}\cdot\nabla f$$

$$\Delta(fg) = f\Delta g + g\Delta f + 2\nabla f \cdot \nabla g$$

$$\Delta(fg) = \frac{\partial^2}{\partial x^2}(fg) + \frac{\partial^2}{\partial y^2}(fg) + \frac{\partial^2}{\partial z^2}(fg)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\cdot g + \frac{\partial g}{\partial x}\cdot f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y} g + \frac{\partial g}{\partial y}\cdot f\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z} g + \frac{\partial g}{\partial z} f\right)$$

$$= \frac{\partial^2 f}{\partial x^2}\cdot g + \frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial^2 g}{\partial x^2} f + \frac{\partial g}{\partial x}\cdot\frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial y^2} g + \frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial y} + \frac{\partial^2 g}{\partial y^2}\cdot f + \frac{\partial g}{\partial y}\cdot\frac{\partial f}{\partial y} + \frac{\partial^2 f}{\partial z^2}\cdot g + \frac{\partial f}{\partial z}\cdot\frac{\partial g}{\partial z} + \frac{\partial^2 g}{\partial z^2} f + \frac{\partial g}{\partial z}\cdot\frac{\partial f}{\partial z}$$

$$= \Delta f \cdot g + \Delta g f + 2\cdot\nabla f \cdot \nabla g$$

$\boxed{P3}$ a) $r(\alpha, \beta) = (\cos(\alpha)\sin(\beta), \sin(\alpha)\sin(\beta), \cos(\beta))$

$$\frac{\partial r}{\partial \alpha} = (-\sin(\alpha)\sin(\beta), \cos(\alpha)\sin(\beta), 0)$$

$$\frac{\partial r}{\partial \beta} = (\cos(\alpha)\cos(\beta), \sin(\alpha)\cos(\beta), -\sin(\beta))$$

$$\frac{\partial r}{\partial \alpha} \times \frac{\partial r}{\partial \beta} = \left|\begin{array}{ccc|cc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ -\sin(\alpha)\sin(\beta) & \cos(\alpha)\sin(\beta) & 0 & -\sin(\alpha)\sin(\beta) & \cos(\alpha)\sin(\beta) \\ \cos(\alpha)\cos(\beta) & \sin(\alpha)\cos(\beta) & -\sin(\beta) & \cos(\alpha)\cos(\beta) & \sin(\alpha)\cos(\beta) \end{array}\right|$$

$$= (-\cos(\alpha)\sin^2(\beta), -\sin(\alpha)\sin(\beta)^2, -\sin(\beta)\cos(\beta))$$

$$\left\|\frac{\partial r}{\partial \alpha}\times\frac{\partial r}{\partial \beta}\right\|^2=\Big(\sin^4(\beta)\underbrace{(\cos^2(\alpha)+\sin^2(\alpha))}_{1}+\sin^2(\beta)\cos^2(\beta)\Big)$$

$$=\sin^2(\beta)\left(\sin^2(\beta)+\cos^2(\beta)\right)$$

$$=\sin^2(\beta)$$

$\Rightarrow \left\|\frac{\partial r}{\partial \alpha}\times\frac{\partial r}{\partial \beta}\right\|=\sin(\beta)$, como $\beta\in[0,\pi]\Rightarrow \sin(\beta)\geq 0$

$\Rightarrow \hat{n}=\left(-\cos(\alpha)\sin(\beta),-\sin(\alpha)\sin(\beta),\cos(\alpha)\right)$

b) Aux 5

c) No es regular, pues para $\beta=0$ $\left\|\frac{\partial r}{\partial \alpha}\times\frac{\partial r}{\partial \beta}\right\|=0$