Asset Pricing

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Chapter 5. Mean-variance frontier and beta representations

Much empirical work in asset pricing is couched in terms of expected return - beta representations and mean-variance frontiers. This chapter introduces expected return - beta representations and mean-variance frontiers.

I discuss here the beta *representation*, most commonly applied to factor pricing models. Chapter 6 shows how an expected return/beta model is equivalent to a linear model for the discount factor, i.e. $m = b'f$ where f are the right hand variables in the time-series regressions that define betas. Chapter 9 then discusses the *derivation* of popular factor models such as the CAPM, ICAPM and APT, i.e. under what assumptions the discount factor *is* a linear function of other variables f such as the market return.

I summarize the classic Lagrangian approach to the mean-variance frontier. I then introduce a powerful and useful representation of the mean-variance frontier due to Hansen and Richard (1987). This representation uses the state-space geometry familiar from the existence theorems. It is also useful because it is valid and useful in infinite-dimensional payoff spaces, which we shall soon encounter when we add conditioning information, dynamic trading or options.

5.1 Expected return - Beta representations

The expected return-beta expression of a factor pricing model is

$$E(R^i) = \alpha + \beta_{i,a}\lambda_a + \beta_{i,b}\lambda_b + \dots$$

The model is equivalent to a restriction that the intercept is the same for all assets in time-series regressions.

When the factors are returns excess returns, then $\lambda_a = E(f^a)$. If the test assets are also excess returns, then the intercept should be zero, $\alpha = 0$.

Much empirical work in finance is cast in terms of expected return - beta representations of linear factor pricing models, of the form

$$E(R^i) = \alpha + \beta_{i,a}\lambda_a + \beta_{i,b}\lambda_b + \dots,\ i = 1, 2, ...N. \tag{55}$$

The β terms are defined as the coefficients in a multiple regression of returns on factors,

$$R_t^i = a_i + \beta_{i,a} f_t^a + \beta_{i,b} f_t^b + \ldots + \varepsilon_t^i; \quad t = 1, 2, ...T. \tag{56}$$

This is often called a *time-series regression*, since one runs a regression across time for each security i. The "factors" f are proxies for marginal utility growth. I discuss the stories used to select factors at some length in chapter 9. For the moment keep in mind the canonical examples, $f =$ consumption growth, or $f =$ the return on the market portfolio (CAPM). Notice that we run returns R_t^i on contemporaneous factors f_t^j. This regression is not about predicting returns from variables seen ahead of time. Its objective is to measure contemporaneous relations or risk exposure; whether returns are typically high in "good times" or "bad times" as measured by the factors.

The point of the beta model(5.55) is to explain the variation in average returns across assets. I write $i = 1, 2, ...N$ in (5.55) to emphasize this fact. The model says that assets with higher betas should get higher average returns. Thus the betas in (5.55) are the explanatory (x) variables, which vary asset by asset. The α and λ – common for all assets – are the intercept and slope in this cross-sectional relation. For example, equation (5.55) says that if we plot expected returns versus betas in a one-factor model, we should expect all $(E(R^i),\ \beta_{i,a})$ pairs to line up on a straight line with slope λ_a and intercept α.

$\beta_{i,a}$ is interpreted as the amount of exposure of asset i to factor a risks, and λ_a is interpreted as the price of such risk-exposure. Read the beta pricing model to say: "for each unit of exposure β to risk factor a, you must provide investors with an expected return premium λ_a." Assets must give investors higher average returns (low prices) if they pay off well in times that are already good, and pay off poorly in times that are already bad, as measured by the factors.

One way to estimate the free parameters (α, λ) and to test the model (5.55) is to run a *cross sectional regression* of average returns on betas,

$$E(R^i) = \alpha + \beta_{i,a}\lambda_a + \beta_{i,b}\lambda_b + \ldots + \alpha_i,\ i = 1, 2, ...N. \tag{57}$$

Again, the β_i are the right hand variables, and the α and λ are the intercept and slope coefficients that we estimate in this cross-sectional regression. The errors α_i are *pricing errors.* The model predicts $\alpha_i = 0$, and they should be statistically insignificant in a test. (I intentionally use the same symbol for the intercept, or mean of the pricing errors, and the individual pricing errors α_i.) In the chapters on empirical technique, we will see test statistics based on the sum of squared pricing errors.

The fact that the betas are regression coefficients is crucially important. If the betas are also free parameters then there is no content to the equation. More importantly (and this is an easier mistake to make), the betas cannot be asset-specific or firm-specific characteristics, such as the size of the firm, book to market ratio, or (to take an extreme example) the letter of the alphabet of its ticker symbol. It is true that expected returns are *associated with* or *correlated with* many such characteristics. Stocks of small companies or of companies with high book/market ratios do have higher average returns. But this correlation must be *explained* by some beta. The proper betas should drive out any characteristics in cross-sectional regressions. If, for example, expected returns were truly related to size, one could buy many small companies to form a large holding company. It would be a "large" company, and hence pay low average returns to the shareholders, while earning a large average return on its holdings. The managers could enjoy the difference. What ruins this promising idea? . The "large" holding company will still *behave* like a portfolio of small stocks. Thus, only if asset returns depend on *how you behave*, not *who you are* – on betas rather than characteristics – can a market equilibrium survive such simple repackaging schemes.

Some common special cases

If there is a risk free rate, its betas in (5.55) are all zero,[2] so the intercept is equal to the risk free rate,

$$R^f = \alpha.$$

We can impose this condition rather than estimate α in the cross-sectional regression (5.57). If there is no risk-free rate, then α must be estimated in the cross-sectional regression. Since it is the expected return of a portfolio with zero betas on all factors, α is called the (expected) *zero-beta rate* in this circumstance.

We often examine factor pricing models using excess returns directly. (There is an implicit, though not necessarily justified, division of labor between models of interest rates and models of equity risk premia.) Differencing (5.55) between any two returns $R^{ei} = R^i - R^j$ (R^j does not have to be risk free), we obtain

$$E(R^{ei}) = \beta_{i,a}\lambda_a + \beta_{i,b}\lambda_b + \dots,\ i = 1, 2, ...N. \tag{58}$$

Here, β_{ia} represents the regression coefficient of the excess return R^{ei} on the factors.

It is often the case that the factors are also returns or excess returns. For example, the CAPM uses the return on the market portfolio as the single factor. In this case, the model should apply to the factors as well, and this fact allows us to directly measure the λ coefficients. Each factor has beta of one on itself and zero on all the other factors, of course. Therefore, if the factors are excess returns, we have $E(f^a) = \lambda_a$, and so forth. We can then write the factor model as

$$E(R^{ei}) = \beta_{i,a}E(f^a) + \beta_{i,b}E(f^b) + \dots,\ \ i = 1, 2, ...N.$$

The cross-sectional beta pricing model (5.55)-(5.58) and the time-series regression definition of the betas in (5.56) look very similar. It seems that one can take expectations of

[2] The betas are zero because the risk free rate is known ahead of time. When we consider the effects of conditioning information, i.e. that the interest rate could vary over time, we have to interpret the means and betas as *conditional* moments. Thus, if you are worried about time-varying risk free rates, betas, and so forth, either assume all variables are i.i.d. (and thus that the risk free rate is constant), or interpret all moments as conditional on time t information.

the time-series regression (5.56) and arrive at the beta model (5.55), in which case the latter would be vacuous since one can always run a regression of anything on anything. The difference is subtle but crucial: the time-series regressions (5.56) will in general have a different intercept a_i for each return i, while the intercept α is the same for all assets in the beta pricing equation (5.55). The beta pricing equation is a restriction on expected returns, and thus imposes a restriction on intercepts in the time-series regression.

In the special case that the factors are themselves excess returns, the restriction is particularly simple: the time-series regression intercepts should all be zero. In this case, we can avoid the cross-sectional regression entirely, since there are no free parameters left.

5.2 Mean-variance frontier: Intuition and Lagrangian characterization

The *mean-variance frontier* of a given set of assets is the boundary of the set of means and variances of the returns on all portfolios of the given assets. One can find or define this boundary by minimizing return variance for a given mean return. Many asset pricing propositions and test statistics have interpretations in terms of the mean-variance frontier.

Figure 13 displays a typical mean-variance frontier. As displayed in Figure 13, it is common to distinguish the mean-variance frontier of all risky assets, graphed as the hyperbolic region, and the mean-variance frontier of all assets, i.e. including a risk free rate if there is one, which is the larger wedge-shaped region. Some authors reserve the terminology "mean-variance frontier" for the upper portion, calling the whole thing the *minimum variance frontier*. The risky asset frontier is a hyperbola, which means it lies between two asymptotes, shown as dotted lines. The risk free rate is typically drawn below the intersection of the asymptotes and the vertical axis, or the point of minimum variance on the risky frontier. If it were above this point, investors with a mean-variance objective would try to short the risky assets, which cannot represent an equilibrium.

In general, portfolios of two assets or portfolios fill out a hyperbolic curve through the two assets. The curve is sharper the less correlated are the two assets, because the portfolio variance benefits from increasing diversification. Portfolios of a risky asset and risk free rate give rise to straight lines in mean-standard deviation space.

In Chapter 1, we derived a similar wedge-shaped region as the set of means and variances of all assets that are priced by a given discount factor. This chapter is about incomplete markets, so we think of a mean-variance frontier generated by a given set of assets, typically less than complete.

When does the mean-variance frontier exist? I.e., when is the set of portfolio means and variances less than the whole $\{E, \sigma\}$ space? We basically have to rule out a special case: two returns are perfectly correlated but yield different means. In that case one could short one, long the other, and achieve infinite expected returns with no risk. More formally, eliminate purely redundant securities from consideration, then

Theorem: So long as the variance-covariance matrix of returns is non-singular, there is a mean-variance frontier.

To prove this theorem, just follow the construction below. This theorem should sound very familiar: Two perfectly correlated returns with different mean are a violation of the law of one price. Thus, the law of one price implies that there is a mean variance frontier as well as a discount factor.

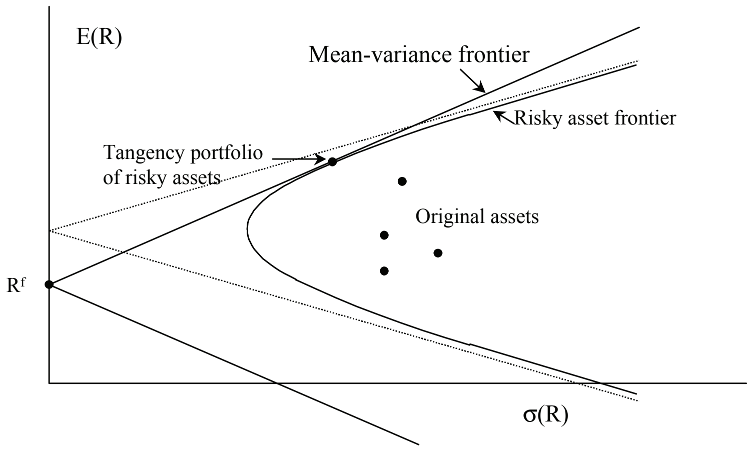

Figure 13. Mean-variance frontier

5.2.1 Lagrangian approach to mean-variance frontier

The standard definition and the computation of the mean-variance frontier follows a brute force approach.

Problem: Start with a vector of asset returns R. Denote by E the vector of mean returns, $E \equiv E(R)$, and denote by Σ the variance-covariance matrix $\Sigma = E\left[(R-E)(R-E)'\right]$. A portfolio is defined by its weights w on the initial securities. The portfolio return is $w'R$ where the weights sum to one, $w'1 = 1$. The problem "choose a portfolio to minimize variance for a given mean" is then

$$min_{\{w\}} \;\; w'\Sigma w \text{ s.t. } w'E = \mu; \; w'1 = 1. \tag{59}$$

Solution: Let

$$A = E'\Sigma^{-1}E; \;\; B = E'\Sigma^{-1}1; \;\; C = 1'\Sigma^{-1}1.$$

Then, for a given mean portfolio return μ, the minimum variance portfolio has variance

$$var\left(R^p\right) = \frac{C\mu^2 - 2B\mu + A}{AC - B^2} \tag{60}$$

and is formed by portfolio weights

$$w = \Sigma^{-1} \frac{E\left(C\mu - B\right) + 1\left(A - B\mu\right)}{\left(AC - B^2\right)}.$$

Equation (5.60) shows that the variance is a quadratic function of the mean. The square root of a parabola is a hyperbola, which is why we draw hyperbolic regions in mean-standard deviation space.

The *minimum-variance portfolio* is interesting in its own right. It appears as a special case in many theorems and it appears in several test statistics. We can find it by minimizing (5.60) over μ, giving $\mu^{\text{min var}} = B/C$. The weights of the minimum variance portfolio are thus

$$w = \Sigma^{-1}1/(1'\Sigma^{-1}1).$$

We can get to any point on the mean-variance frontier by starting with two returns on the frontier and forming portfolios. The frontier is *spanned* by any two frontier returns. To see this fact, notice that w is a linear function of μ. Thus, if you take the portfolios corresponding to any two distinct mean returns μ_1 and μ_2, the weights on a third portfolio with mean $\mu_3 = \lambda\mu_1 + (1-\lambda)\mu_2$ are given by $w_3 = \lambda w_1 + (1-\lambda)w_2$.

Derivation: To derive the solution, introduce Lagrange multipliers 2λ and 2δ on the constraints. The first order conditions to (5.59) are then

$$\Sigma w - \lambda E - \delta 1 = 0$$

$$w = \Sigma^{-1}(\lambda E + \delta 1). \tag{61}$$

We find the Lagrange multipliers from the constraints,

$$E'w = E'\Sigma^{-1}(\lambda E + \delta 1) = \mu$$

$$1'w = 1'\Sigma^{-1}(\lambda E + \delta 1) = 1$$

or

$$\begin{bmatrix} E'\Sigma^{-1}E & E'\Sigma^{-1}1 \\ 1'\Sigma^{-1}E & 1'\Sigma^{-1}1 \end{bmatrix} \begin{bmatrix} \lambda \\ \delta \end{bmatrix} = \begin{bmatrix} \mu \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} \lambda \\ \delta \end{bmatrix} = \begin{bmatrix} \mu \\ 1 \end{bmatrix}$$

Hence,

$$\lambda = \frac{C\mu - B}{AC - B^2}$$

$$\delta = \frac{A - B\mu}{AC - B^2}$$

Plugging in to (5.61), we get the portfolio weights and variance.

5.3 An orthogonal characterization of the mean-variance frontier

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Every return can be expressed as $R^i = R^* + w^i R^{e*} + n^i$.

The mean-variance frontier is $R^{mv} = R^* + wR^{e*}$

R^{e*} is defined as $R^{e*} = proj(1|\underline{R^e})$. It represents mean excess returns, $E(R^e) = E(R^{e*}R^e)$ $\forall R^e \in \underline{R^e}$

The Lagrangian approach to the mean-variance frontier is straightforward but cumbersome. Our further manipulations will be easier if we follow an alternative approach due to Hansen and Richard (1987). Technically, Hansen and Richard's approach is also valid when we can't generate the payoff space by portfolios of a finite set of basis payoffs $c'x$. This happens, for example, when we think about conditioning information in Chapter 8. Also, it is the natural geometric way to think about the mean-variance frontier given that we have started to think of payoffs, discount factors and other random variables as vectors in the space of payoffs. Rather than write portfolios as combinations of basis assets, and pose and solve a minimization problem, we first describe any return by a three-way orthogonal decomposition. The mean-variance frontier then pops out easily without any algebra.

5.3.1 Definitions of R^*, R^{e*}

I start by defining two special returns. R^* is the return corresponding to the payoff x^* that can act as the discount factor. The price of x^*, is, like any other price, $p(x^*) = E(x^*x^*)$. Thus,

The definition of R^* *is*

$$R^* \equiv \frac{x^*}{p(x^*)} = \frac{x^*}{E(x^{*2})} \tag{62}$$

The definition of R^{e*} *is*

$$R^{e*} \equiv proj(1 \mid \underline{R}^e) \tag{63}$$

$$\underline{R}^e \equiv \text{space of excess returns} = \{x \in \underline{X} \ s.t. \ p(x) = 0\}$$

Why R^{e*}? We are heading towards a mean-variance frontier, so it is natural to seek a special return that changes means. R^{e*} is an excess return that represents means on $\underline{R}^e$ with an inner product in the same way that x^* is a payoff in $\underline{X}$ that represents prices with an inner product. As

$$p(x) = E(mx) = E[proj(m|\underline{X})x] = E(x^* x),$$

so

$$E(R^e) = E(1 \times R^e) = E\left[proj(1 \mid R^e) \times R^e\right] = E(R^{e*} R^e).$$

If R^* and R^{e*} are still a bit mysterious at this point, they will make more sense as we use them, and discover their many interesting properties.

Now we can state a beautiful orthogonal decomposition.

Theorem: Every return R^i can be expressed as

$$R^i = R^* + w^i R^{e*} + n^i$$

where w^i is a number, and n^i is an excess return with the property

$$E(n^i) = 0.$$

The three components are orthogonal,

$$E(R^* R^{e*}) = E(R^* n^i) = E(R^{e*} n^i) = 0.$$

This theorem quickly implies the characterization of the mean variance frontier which we are after:

Theorem: R^{mv} is on the mean-variance frontier if and only if

$$R^{mv} = R^* + wR^{e*} \tag{64}$$

for some real number w.

As you vary the number w, you sweep out the mean-variance frontier. $E(R^{e*}) \neq 0$, so adding more w changes the mean and variance of R^{mv}. You can interpret (5.64) as a "twofund" theorem for the mean-variance frontier. It expresses every frontier return as a portfolio of R^* and R^{e*}, with varying weights on the latter.

As usual, first I'll argue why the theorems are sensible, then I'll offer a simple algebraic proof. Hansen and Richard (1987) give a much more careful algebraic proof.

5.3.2 Graphical construction

Figure 14 illustrates the decomposition. Start at the origin (0). Recall that the x^* vector is perpendicular to planes of constant price; thus the R^* vector lies perpendicular to the plane of returns as shown. Go to R^*.

R^{e*} is the excess return that is closest to the vector 1; it lies at right angles to planes (in $\underline{R^e}$) of constant *mean* return, shown in the $E = 1, E = 2$ lines, just as the return R^* lies at right angles to planes of constant price. Since R^{e*} is an excess return, it is orthogonal to R^*. Proceed an amount w^i in the direction of R^{e*}, getting as close to R^i as possible.

Now move, again in an orthogonal direction, by an amount n^i to get to the return R^i. We have thus expressed $R^i = R^* + w^i R^{e*} + n^i$ in a way that all three components are orthogonal.

Returns with $n = 0$, $R^* + wR^{e*}$, are the mean-variance frontier. Here's why. Since $E(R^2) = \sigma^2(R) + E(R)^2$, we can define the mean-variance frontier by minimizing second moment for a given mean. The length of each vector in Figure 14 is its second moment, so we want the shortest vector that is on the return plane for a given mean. The shortest vectors in the return plane with given mean are on the $R^* + wR^{e*}$ line.

The graph also shows how R^{e*} represents means in the space of excess returns. Expectation is the inner product with 1. Planes of constant expected value in Figure 14 are perpendicular to the 1 vector, just as planes of constant price are perpendicular to the x^* or R^* vectors. I don't show the full extent of the constant expected payoff planes for clarity; I do show lines of constant expected excess return in $\underline{R^e}$, which are the intersection of constant expected payoff planes with the $\underline{R^e}$ plane. Therefore, just as we found an x^* in $\underline{X}$ to represent prices in $\underline{X}$ by projecting m onto $\underline{X}$, we find R^{e*} in $\underline{R^e}$ by projecting of 1 onto $\underline{R^e}$. Yes, a regression with one on the left hand side. Planes perpendicular to R^{e*} in $\underline{R^e}$ are payoffs with constant *mean*, just as planes perpendicular to x^* in $\underline{X}$ are payoffs with the same *price*.

5.3.3 Algebraic argument

Now, an algebraic proof of the decomposition and characterization of mean variance frontier. The algebra just represents statements about what is at right angles to what with second moments.

Proof: Straight from their definitions, (5.62) and (5.63) we know that R^{e*} is an

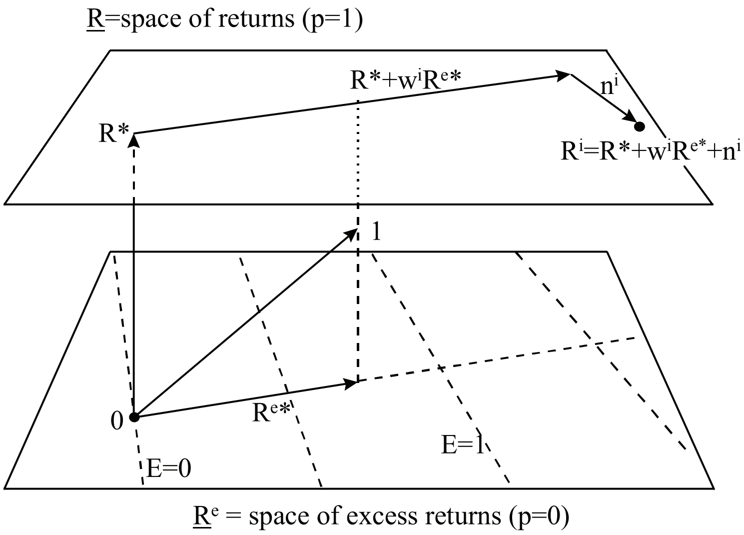

Figure 14. Orthogonal decomposition and mean-variance frontier.

excess return (price zero), and hence that R^* and R^{e*} are orthogonal,

$$E(R^*R^{e*}) = \frac{E(x^*R^{e*})}{E(x^{*2})} = 0.$$

We define n^i so that the decomposition adds up to R^i as claimed, and we define w^i to make sure that n^i is orthogonal to the other two components. Then we prove that $E(n^i) = 0$. Pick any w^i and then define

$$n^i \equiv R^i - R^* - w^i R^{e*}.$$

n^i is an excess return so already orthogonal to R^*,

$$E(R^*n^i) = 0.$$

To show $E(n^i) = 0$ and n^i orthogonal to R^{e*}, we exploit the fact that since n^i is an excess return,

$$E(n^i) = E(R^{e*}n^i).$$

Therefore, R^{e*} is orthogonal to n^i if and only if we pick w^i so that $E(n^i) = 0$. We

don't have to explicitly calculate w^i for the proof.[3]
Once we have constructed the decomposition, the frontier drops out. Since $E(n^i) = 0$ and the three components are orthogonal,

$$E(R^i) = E(R^*) + w^i E(R^{e*})$$

$$\sigma^2(R^i) = \sigma^2(R^* + w^i R^{e*}) + \sigma^2(n^i).$$

Thus, for each desired value of the mean return, there is a unique w^i. Returns with $n^i = 0$ minimize variance for each mean. ■

5.3.4 Decomposition in mean-variance space

Figure 15 illustrates the decomposition in mean-variance space rather than in state-space.

First, let's locate R^*. R^* is the minimum second moment return. One can see this fact from the geometry of Figure 14: R^* is the return closest to the origin, and thus the return with the smallest "length" which is second moment. As with OLS regression, minimizing the length of R^* and creating an R^* orthogonal to all excess returns is the same thing. One can also verify this property algebraically. Since any return can be expressed as $R = R^* + wR^{e*} + n$, $E(R^2) = E(R^{*2}) + w^2 E(R^{e*2}) + E(n^2)$. $n = 0$ and $w = 0$ thus give the minimum second moment return.

In mean-standard deviation space, lines of constant second moment are circles. Thus, the minimum second-moment return R^* is on the smallest circle that intersects the set of all assets, which lie in the mean-variance frontier in the right hand panel of Figure 19. Notice that R^* is on the lower, or "inefficient" segment of the mean-variance frontier. It is initially surprising that this is the location of the most interesting return on the frontier! R^* is *not* the "market portfolio" or "wealth portfolio," which typically lie on the upper portion of the frontier.

Adding more R^{e*} moves one along the frontier. Adding n does not change mean but does change variance, so it is an *idiosyncratic* return that just moves an asset off the frontier as graphed. α is the "zero-beta rate" corresponding to R^*. It is the expected return of any return that is uncorrelated with R^*. I demonstrate these properties in section 6.5.

[3] Its value

$$w^i = \frac{E(R^i) - E(R^*)}{E(R^{e*})}$$

is not particularly enlightening.

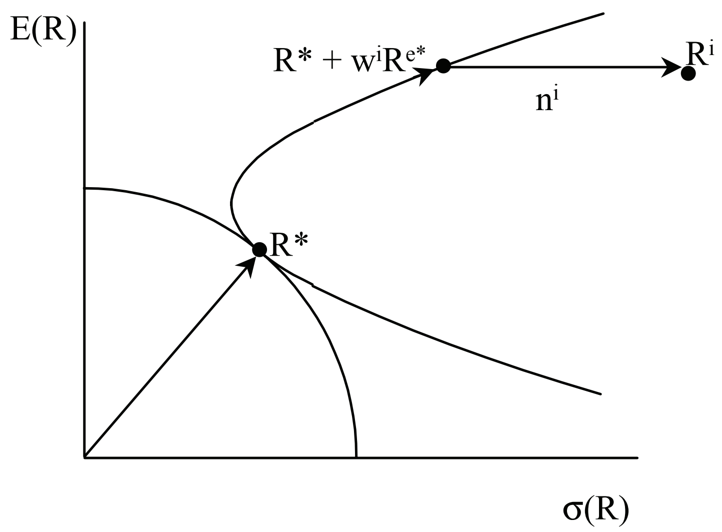

Figure 15. Orthogonal decomposition of a return R^i in mean-standard deviation space.

5.4 Spanning the mean-variance frontier

The characterization of the mean-variance frontier in terms of R^* and R^{e*} is most natural in our setup. However, you can equivalently span the mean-variance frontier with any two portfolios that are on the frontier – any two distinct linear combinations of R^* and R^{e*}. In particular, take any return

$$R^\alpha = R^* + \gamma R^{e*}, \quad \gamma \neq 0. \tag{65}$$

Using this return in place of R^{e*},

$$R^{e*} = \frac{R^\alpha - R^*}{\gamma}$$

you can express the mean variance frontier in terms of R^* and R^α :

$$\begin{aligned} R^* + wR^{e*} &= R^* + y\left(R^\alpha - R^*\right) \\ &= (1-y)R^* + yR^\alpha \end{aligned} \tag{5.66}$$

where I have defined a new weight $y = w/\gamma$.

The most common alternative approach is to use a risk free rate or a risky rate that somehow behaves like the risk free rate in place of R^{e*} to span the frontier. When there is a risk free rate, it is on the frontier with representation

$$R^f = R^* + R^f R^{e*}$$

I derive this expression in equation (5.72) below. Therefore, we can use (5.66) with $R^a = R^f$. When there is no risk free rate, several risky returns that retain some properties of the risk free rate are often used. In section 5.3 below I present a "zero beta" return, which is uncorrelated with R^*, a "constant-mimicking portfolio" return, which is the return on the traded payoff closest to unity, $\hat{R} = proj(1|\underline{X})/p[proj(1|\underline{X})]$ and the minimum variance return. Each of these returns is on the mean-variance frontier, with form 5.65, though different values of α. Therefore, we can span the mean-variance frontier with R^* and any of these risk-free rate proxies.

5.5 A compilation of properties of R^*, R^{e*} and x^*

The special returns R^*, R^{e*} that generate the mean variance frontier have lots of interesting and useful properties. Some I derived above, some I will derive and discuss below in more detail, and some will be useful tricks later on. Most properties and derivations are extremely obscure if you don't look at the picture!

1)

$$E(R^{*2}) = \frac{1}{E(x^{*2})}. \tag{67}$$

To derive this fact, multiply both sides of (5.62) by R^*, take expectations, and remember R^* is a return so $1 = E(x^* R^*)$.

2) We can reverse the definition and recover x^* from R^* via

$$x^* = \frac{R^*}{E(R^{*2})}. \tag{68}$$

To derive this formula, start with the definition $R^* = x^*/E(x^{*2})$ and substitute from (5.67) for $E(x^{*2})$

3) R^* can be used to represent prices just like x^*. This is not surprising, since they both point in the same direction, orthogonal to planes of constant price. Most obviously, from 5.68

$$p(x) = E(x^* x) = \frac{E(R^* x)}{E(R^{*2})} \; \forall x \in \underline{X}$$

For returns, we can nicely express this result as

$$E(R^{*2}) = E(R^* R) \; \forall R \in \underline{R}. \tag{69}$$

This fact can also serve as an alternative defining property of R^*.

4) R^{e*} represents means on $\underline{R^e}$ via an inner product in the same way that x^* represents prices on $\underline{X}$ via an inner product. R^{e*} is orthogonal to planes of constant mean in $\underline{R^e}$ as x^* is orthogonal to planes of constant price. Algebraically, in analogy to $p(x) = E(x^* x)$ we have

$$E(R^e) = E(R^{e*} R^e)\ \forall R^e \in \underline{R^e}. \tag{70}$$

This fact can serve as an alternative defining property of R^{e*}.

5) R^{e*} and R^* are orthogonal,

$$E(R^* R^{e*}) = 0.$$

More generally, R^* is orthogonal to any excess return.

6) The mean variance frontier is given by

$$R^{mv} = R^* + wR^{e*}.$$

To prove this, $E(R^2) = E\left[(R^* + wR^{e*} + n)^2\right] = E(R^{*2}) + w^2 E(R^{e2}) + E(n^2)$, and $E(n) = 0$, so set n to zero. The conditional mean-variance frontier allows w in the conditioning information set. The unconditional mean variance frontier requires w to equal a constant.

7) R^* is the minimum second moment return. Graphically, R^* is the return closest to the origin. To see this, using the decomposition in #6, and set w^2 and n to zero to minimize second moment.

8) R^{e*} has the same first and second moment,

$$E(R^{e*}) = E(R^{e*2}).$$

Just apply fact (5.70) to R^{e*} itself. Therefore

$$var(R^{e*}) = E(R^{e*2}) - E(R^{e*})^2 = E(R^{e*})\left[1 - E(R^{e*})\right].$$

9) If there is a riskfree rate, then R^{e*} can also be defined as the residual in the projection of 1 on R^* :

$$R^{e*} = 1 - proj(1|R^*) = 1 - \frac{E(R^*)}{E(R^{*2})} R^* = 1 - \frac{1}{R^f} R^* \tag{71}$$

You'd never have thought of this without looking at Figure 14! Since R^* and $\underline{R^e}$ are orthogonal and together span $\underline{X}$, $1 = proj(1|\underline{R^e}) + proj(1|R^*)$. You can also verify this statement analytically. Check that R^{e*} so defined is an excess return in $\underline{X}$ – its price is zero–, and $E(R^{e*} R^e) = E(R^e)$; $E(R^* R^{e*}) = 0$.

As a result, R^f has the decomposition

$$R^f = R^* + R^f R^{e*}. \tag{72}$$

Since $R^f > 1$ typically, this means that $R^* + R^{e*}$ is located on the lower portion of the mean-variance frontier in mean-variance space, just a bit to the right of R^f. If the risk free rate were one, then the unit vector would lie in the return space, and $R^f = R^* + R^{e*}$. Typically, the space of returns is a little bit above the unit vector. As you stretch the unit vector by the amount R^f to arrive at the return R^f, so you stretch the amount R^{e*} that you add to R^* to get to R^f.

If there is no riskfree rate, then we can use

$$\begin{aligned} proj(1|\underline{X}) &= proj(proj\,(1|\underline{X})\,|\underline{R^e}) + proj(proj\,(1|\underline{X})\,|\underline{R^*}) \\ &= proj(1|\underline{R^e}) + proj(1|\underline{R^*}) \end{aligned}$$

to deduce an analogue to equation (5.71),

$$R^{e*} = proj(1|X) - proj(1|R^*) = proj(1|X) - \frac{E(R^*)}{E(R^{*2})} R^* \tag{73}$$

10) If a riskfree rate is traded, we can construct R^f from R^* via

$$R^f = \frac{1}{E(x^*)} = \frac{E(R^{*2})}{E(R^*)}. \tag{74}$$

If not, this gives a “zero beta rate” interpretation of the right hand expression.

11) Since we have a formula $x^* = p'E(xx')^{-1}x$ for constructing x^* from basis assets (see section 4.1), we can construct R^* in this case from

$$R^* = \frac{x^*}{p(x^*)} = \frac{p'E(xx')^{-1}x}{p'E(xx')^{-1}p}.$$

($p(x^*) = E(x^* x^*)$ leading to the denominator.)

12) We can construct R^{e*} from a set of basis assets as well. Following the definition to project one on the space of excess returns,

$$R^{e*} = E(R^e)'E(R^e R^{e\prime})^{-1} R^e$$

where R^e is the basis set of excess returns. (You can always use $R^e = R - R^*$ if you want). This construction obviously mirrors the way we constructed x^* in section 4.1, and you can see the similarity in the result, with E in place of p, since R^{e*} represents means rather than prices. .

If there is a riskfree rate, we can also use (5.71),

$$R^{e*} = 1 - \frac{1}{R^f}R^* = 1 - \frac{1}{R^f}\frac{p'E(xx')^{-1}x}{p'E(xx')^{-1}p}. \tag{75}$$

If there is no riskfree rate, we can use (5.73) to construct R^{e*}. The central ingredient is

$$proj(1|\underline{X}) = E(x)'E(xx')^{-1}x.$$

5.6 Mean-variance frontiers for m: the Hansen-Jagannathan bounds

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The mean-variance frontier of all discount factors that price a given set of assets is related to the mean-variance frontier of asset returns by

$$\frac{\sigma(m)}{E(m)} \geq \frac{|E(R^e)|}{\sigma(R^e)}.$$

and hence

$$\min_{\{\text{all } m \text{ that price } x\in\underline{X}\}} \frac{\sigma(m)}{E(m)} = \max_{\{\text{all excess returns } R^e \text{ in } \underline{X}\}} \frac{E(R^e)}{\sigma(R^e)}$$

The discount factors on the frontier can be characterized analogously to the mean-variance frontier of asset returns,

$$\begin{aligned} m &= x^* + we^* \\ e^* &\equiv 1 - proj(1|\underline{X}) = proj(1|\underline{E}) = 1 - E(x)'E(xx')^{-1}x \\ \underline{E} &= \{m - x^*\}. \end{aligned}$$

We derived in Chapter 1 a relation between the Sharpe ratio of an excess return and the volatility of discount factors necessary to price that return,

$$\frac{\sigma(m)}{E(m)} \geq \frac{|E(R^e)|}{\sigma(R^e)}.$$

Quickly,

$$0 = E(mR^e) = E(m)E(R^e) + \rho_{m,R^e}\sigma(m)\sigma(R^e),$$

and $|\rho| \leq 1$. If we had a riskfree rate, then we know in addition

$$E(m) = 1/R^f.$$

Hansen and Jagannathan (1991) had the brilliant insight to read this equation as a restriction on the set of *discount factors* that can price a given set of returns, as well as a restriction on the set of *returns* we will see given a specific discount factor. This calculation teaches us that we need very volatile discount factors with a mean near one to understand stock returns. This and more general related calculations turn out to be a central tool in understanding and surmounting the equity premium puzzle, surveyed in Chapter 21.

We would like to derive a bound that uses a large number of assets, and that is valid if there is no riskfree rate. What is the set of $\{E(m), \sigma(m)\}$ consistent with a given set of asset prices and payoffs? What is the mean-variance frontier for discount factors?

Obviously, the higher the Sharpe ratio, the tighter the bound. This suggests a way to construct the frontier we're after. For any hypothetical risk-free rate, find the highest Sharpe ratio. That is, of course the tangency portfolio. Then the slope to the tangency portfolio gives the ratio $\sigma(m)/E(m)$. Figure 16 illustrates.

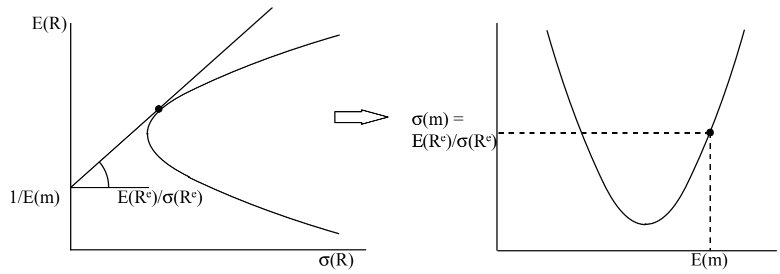

Figure 16. Graphical construction of the Hansen-Jagannathan bound.

As we sweep through values of $E(m)$, the slope to the tangency becomes lower, and the Hansen-Jagannathan bound declines. At the mean return corresponding to the minimum variance point, the HJ bound attains its minimum. Continuing, the Sharpe ratio rises again and so does the bound. If there were a riskfree rate, then we know $E(m)$, the return frontier is a V shape, and the HJ bound is purely a bound on variance.

This discussion implies a beautiful duality between discount factor volatility and Sharpe ratios.

$$\min_{\{\text{all } m \text{ that price } x \in \underline{X}\}} \frac{\sigma(m)}{E(m)} = \max_{\{\text{all excess returns } R^e \text{ in } \underline{X}\}} \frac{E(R^e)}{\sigma(R^e)}. \tag{76}$$

We need formulas for an explicit calculation. In equation (), we found a representation for the set of discount factors that price a given set of asset returns – that satisfy $p = E(mx)$:

$$m = E(m) + [p - E(m)E(x)] \Sigma^{-1} [x - E(x)] + \varepsilon \tag{77}$$

where $\Sigma \equiv cov(x, x')$ and $E(\varepsilon) = 0$, $E(\varepsilon x) = 0$. You can think of this as a regression or projection of any discount factor on the space of payoffs, plus an error. Since $\sigma^2(\varepsilon) > 0$, this representation leads immediately to an explicit expression for the Hansen-Jagannathan bound,

$$\sigma^2(m) \geq (p - E(m)E(x))' \Sigma^{-1} (p - E(m)E(x)) . \tag{78}$$

As all asset returns must lie in a cup-shaped region in $\{E(R), \sigma(R)\}$ space, all discount factors must lie in a parabolic region in $\{E(m), \sigma^2(m)\}$ space, as illustrated in the right hand panel of Figure 16.

We would like an expression for the discount factors on the bound, as we wanted an expression for the returns on the mean variance frontier instead of just a formula for the means and variances. As we found a three way decomposition of all returns, two of which generated the mean-variance frontier of returns, so we can find a three way decomposition of discount factors, two of which generate the mean-variance frontier of discount factors (5.78). I illustrate the construction in Figure 17.

Any m must line in the plane marked $\underline{M}$, perpendicular to $\underline{X}$ through x^*. Any m must be of the form

$$m = x^* + we^* + n.$$

Here, I have just broken up the residual ε in the familiar representation $m = x^* + \varepsilon$ into two components. e^* is defined as the residual from the projection of 1 onto $\underline{X}$ or, equivalently the projection of 1 on the space $\underline{E}$ of "excess m's," random variables of the form $m - x^*$.

$$e^* \equiv 1 - proj(1|\underline{X}) = proj(1|\underline{E}).$$

e^* generates means of m just as R^{e*} did for returns:

$$E(m - x^*) = E[1 \times (m - x^*)] = E[proj(1|\underline{E})(m - x^*)].$$

Finally n, defined as the leftovers, has mean zero since it's orthogonal to 1 and is orthogonal to $\underline{X}$.

As with returns, then, the mean-variance frontier of m's is given by

$$m^* = x^* + we^*. \tag{79}$$

If the unit payoff is in the payoff space, then we know $E(m)$, and the frontier and bound are just $m = x^*$, $\sigma^2(m) \geq \sigma^2(x^*)$. This is exactly like the case of risk-neutrality for return mean-variance frontiers.

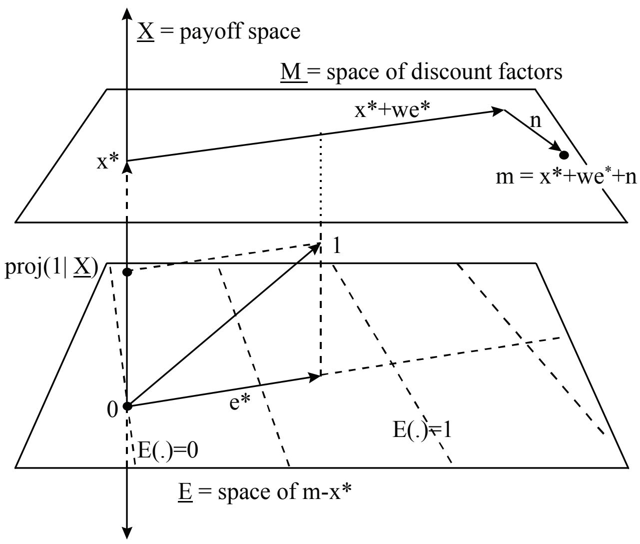

Figure 17. Decomposition of any discount factor $m = x^* + we + n$.

The construction (5.79) can be used to derive the formula (5.78) for the Hansen-Jagannathan bound for the finite-dimensional cases discussed above. It's more general, since it can be used in infinite-dimensional payoff spaces as well. Along with the corresponding return formula $R^{mv} = R^* + wR^{e*}$, we see in Chapter 8 that it extends more easily to the calculation of conditional vs. unconditional mean-variance frontiers (Gallant, Hansen and Tauchen 1995).

It will make construction (5.79) come alive if we find equations for its components. We find x^* as before, it is the portfolio $c'x$ in $\underline{X}$ that prices x:

$$x^* = p'E(xx')^{-1}x.$$

Similarly, let's find e^*. The projection of 1 on $\underline{X}$ is

$$proj(1|\underline{X}) = E(x)'E(xx')^{-1}x.$$

(After a while you get used to the idea of running regressions with 1 on the left hand side and random variables on the right hand side!) Thus,

$$e^* = 1 - E(x)'E(xx')^{-1}x.$$

Again, you can construct time-series of $x^*$and e^* from these definitions.

Finally, we now can construct the variance-minimizing discount factors

$$m^* = x^* + we^* = p'E(xx')^{-1}x + w\left[1 - E(x)'E(xx')^{-1}x\right]$$

or

$$m^* = w + [p - wE(x)]'\,E(xx')^{-1}x \tag{80}$$

As w varies, we trace out discount factors m^*on the frontier with varying means and variances. It's easiest to find mean and second moment:

$$E(m^*) = w + [p - wE(x)]'\,E(xx')^{-1}E(x)$$

$$E(m^{*2}) = [p - wE(x)]'\,E(xx')^{-1}\ [p - wE(x)]\,;$$

variance follows from $\sigma^2(m) = E(m^2) - E(m)^2$. With a little algebra one can also show that these formulas are equivalent to equation (5.78).

As you can see, Hansen-Jagannathan frontiers are equivalent to mean-variance frontiers. For example, an obvious exercise is to see how much the addition of assets raises the Hansen-Jagannathan bound. This is *exactly* the same as asking how much those assets expand the mean-variance frontier. It was, in fact, this link between Hansen-Jagannathan bounds and mean-variance frontiers rather than the logic I described that inspired Knez and Chen (1996) and DeSantis (1994) to test for mean-variance efficiency using, essentially, Hansen-Jagannathan bounds.

Hansen-Jagannathan bounds have the potential to do more than mean-variance frontiers. Hansen and Jagannathan show how to solve the problem

$$\min \sigma^2(m) \text{ s.t. } p = E(mx), m > 0.$$

This is the "Hansen-Jagannathan bound with positivity," and is strictly tighter than the Hansen-Jagannathan bound. It allows you to impose no-arbitrage conditions. In stock applications, this extra bound ended up not being that informative. However, in the option application of this idea of Chapter (18), positivity is really important. That chapter shows how to solve for a bound with positivity.

Hansen, Heaton and Luttmer (1995) develop a distribution theory for the bounds. Luttmer (1996) develops bounds with market frictions such as short-sales constraints and bid-ask spreads, to account for ludicrously high apparent Sharpe ratios and bounds in short term bond data. Cochrane and Hansen (1992) survey a variety of bounds, including bounds that incorporate information that discount factors are poorly correlated with stock returns (the HJ bounds use the extreme $\rho = 1$), bounds on conditional moments that illustrate how many models imply excessive interest rate variation, bounds with short-sales constraints and market frictions, etc.

Chapter 21 discusses what the results of Hansen Jagannathan bound calculations and what they mean for discount factors that can price stock and bond return data.

5.7 Problems

1. Prove that R^{e*} lies at right angles to planes (in $\underline{R^e}$) of constant *mean* return, as shown in figure 14.
2. Should we typically draw x^* above, below or on the plane of returns? *Must* x^* always lie in this position?
3. Can you construct R^{e*} from knowledge of m, x^*, or R^*?
4. What happens to R^*, R^{e*} and the mean-variance frontier if investors are risk neutral?
   (a) If a riskfree rate is traded.
   (b) If no riskfree rate is traded?
   (Hint: make a drawing or think about the case that payoffs are generated by an N dimensional vector of basis assets x)
5. $x^* = proj(m|\underline{X})$. Is $R^* = proj(m|\underline{R})$?
6. We showed that all m are of the form $x^* + \varepsilon$. What about $R^{-1}R$?
7. Show that if there is a risk-free rate—if the unit payoff is in the payoff space $\underline{X}$—then $R^{e*} = (R^f - R^*)/R^f$.