Lucas González M. Aux #1 22/8/22

P1

a) Nos piden normalizar la función de onda, esto significa imponer lo siguiente: $\langle \psi | \psi \rangle = 1 = \int_{-\infty}^{\infty} \langle \psi | x \rangle \langle x | \psi \rangle dx = \int_{-\infty}^{\infty} \Psi^*(x,0)\, \Psi(x,0)\, dx$

$$= \int_{-\infty}^{\infty} \frac{A^2}{(x^2+a^2)^2}\, dx = 1$$

Nos queda por resolver la integral, usamos: $x = a\tan u \Rightarrow dx = a\sec^2 u\, du.$

$$I = \int_{-\pi/2}^{\pi/2} \frac{a\sec^2 du}{(a^2\tan^2 u + a^2)^2} = \frac{a}{a^4}\int_{-\pi/2}^{\pi/2} \frac{\sec^2 u}{\sec^4 u}\, du = \frac{1}{a^3}\int_{-\pi/2}^{\pi/2} \cos^2 u\, du.$$

pero $\cos^2 u - \sin^2 u = \cos(2u) \Leftrightarrow \cos^2 u = \cos 2u + (1 - \cos^2 u) \Leftrightarrow 2\cos^2 u = 1 + \cos 2u$

$\Rightarrow \cos^2 u = \frac{1}{2}(1+\cos 2u)$ reemplazamos en la integral: $\frac{A^2}{a^3}\left[\int_{-\pi/2}^{\pi/2} \frac{1}{2}\, du + \cancel{\int_{-\pi/2}^{\pi/2} \cos(2u)\, du}^{\,0}\right] \overset{!}{=} 1$

$$\Rightarrow \quad A = \sqrt{\frac{2a^3}{\pi}}$$

(b) Veremos como se calcula cada uno de los términos solicitados:

i.- $\langle x \rangle = \langle \psi | x | \psi \rangle = \int_{-\infty}^{\infty} \Psi^*(x,0)\, x\, \Psi(x,0)\, dx = \int_{-\infty}^{\infty} \frac{A^2}{(x^2+a^2)^2} x\, dx$, esto corresponde a una integral impar en dominio simétrico

$\langle x \rangle = 0$

ii) $\langle x^2 \rangle = \int_{-\infty}^{\infty} \frac{A^2 x^2}{(x^2+a^2)^2}\, dx = A^2 \int_{-\infty}^{\infty} \frac{x^2+a^2}{(x^2+a^2)^2} - \frac{a^2}{(x^2+a^2)^2}\, dx = \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2} - \frac{a^2}{(x^2+a^2)^2}\, dx = A\left[\arctan\left(\frac{x}{a}\right)\cdot \frac{1}{a}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}\, dx\right] = \ldots = \frac{A^2\pi}{a^2}$

para calcular la segunda integral se usa $\int \frac{1}{(ax^n+b)^n}\, dx = \frac{2n-3}{2b(n-1)}\int \frac{1}{(ax^2+b)^{n-1}}\, dx + \frac{x}{2b(n-1)(ax^2+b)^{n-1}}$

iii) $\Delta x \equiv \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\langle x^2 \rangle} = \sqrt{\frac{A^2\pi}{a^2}}.$

P2

$$\Psi(x,t) = A e^{-a\left(\frac{mx^2}{\hbar} + it\right)}$$

a) $\langle \psi | \psi \rangle \overset{!}{=} 1 = \int_{-\infty}^{\infty} \Psi^*(x,t)\, \Psi(x,t)\, dx = \int_{-\infty}^{\infty} |A|^2 e^{-\frac{2amx^2}{\hbar}}\, dx = A^2 \int_{-\infty}^{\infty} e^{-x^2/\sqrt{\hbar/2ma}^{\,2}}\, dx$

integral que tiene la forma gaussiana que sabemos $\int_{-\infty}^{\infty} e^{-\frac{x^2}{\sigma^2}}\, dx = \sqrt{\pi}\sigma$ con $\sigma = \sqrt{\frac{\hbar}{2ma}}$

$\Rightarrow \quad 1 = A^2\sqrt{\frac{\pi\hbar}{2am}} \Rightarrow A = \left(\frac{\pi\hbar}{2am}\right)^{\frac{1}{4}}$ //

b) Queremos $\phi(k,0)$, para eso debemos calcular la transformada de Fourier de $\psi(x,0)$:

$$\phi(k)=\int_{-\infty}^{\infty} A e^{-\frac{amx^2}{\hbar}} e^{-kx}\,dx = A\int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{\sigma^2}+kx\right)}dx = A\int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sigma}+\frac{\sigma}{2}k\right)^2-\frac{\sigma^2}{4}k^2}dx$$

$$= Ae^{-\frac{\sigma^2}{4}k^2}\int_{-\infty}^{\infty} e^{-\frac{\left(x+\frac{\sigma^2}{2}k\right)^2}{\sigma^2}}dx = Ae^{-\frac{\sigma^2}{4}k^2}\sqrt{\pi\sigma} \quad //$$

Ojo! A es el mismo que encontramos en (a), no es necesario normalizar de nuevo

c) $\langle x\rangle = \int_{-\infty}^{\infty} x|\psi|^2dx = \int_{-\infty}^{\infty} x A^2 e^{-\frac{2amx^2}{\hbar}}dx = 0$ integrando impar en dominio simétrico.

$$\langle x^2\rangle = \int_{-\infty}^{\infty} x^2|\psi|^2dx = A^2\int_{-\infty}^{\infty} x^2 e^{-\frac{2amx^2}{\hbar}}dx$$

$$\left[\text{trucazo}: \int_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}} \Big/ \frac{d}{da} \Rightarrow \int_{-\infty}^{\infty} -x^2 e^{-ax^2}dx = \sqrt{\pi}\cdot\left(-\tfrac{1}{2}\right)a^{-3/2} \Rightarrow \int_{-\infty}^{\infty} x^2 e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}\right]$$

$$\langle x^2\rangle = A^2\cdot\frac{1}{2}\sqrt{\pi\left(\frac{\hbar}{2am}\right)^3} = A^2\cdot\frac{1}{2}\frac{1}{(2am/\hbar)}\sqrt{\frac{\pi\hbar}{2am}} = \frac{\hbar}{4am} \quad //$$

$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = 0$; Si no tuviese la función en el tiempo esto no se puede hacer, habría que calcular $\langle\psi|p|\psi\rangle = \langle\psi|-i\hbar\frac{d}{dx}|\psi\rangle$

$$\langle p^2\rangle = \int\psi^*\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2\psi\,dx = -\hbar^2\int\psi^*\frac{\partial^2\psi}{\partial x^2}dx = -\hbar^2\int_{-\infty}^{\infty}\psi^*\left[-\frac{2am}{\hbar}\left(1-\frac{2amx^2}{\hbar}\right)\psi\right]dx$$

$$= 2am\hbar\left[\int|\psi|^2dx - \frac{2am}{\hbar}\int x^2|\psi|^2dx\right] = 2am\hbar\left(1-\frac{2am}{\hbar}\langle x^2\rangle\right) = 2am\hbar\left(1-\frac{2am}{\hbar}\cdot\frac{\hbar}{4am}\right) = \dots = am\hbar$$

d) $\sigma_x^2 = \Delta x = \langle x^2\rangle - \langle x\rangle^2 = \frac{\hbar}{4am} \Rightarrow \sigma_x = \sqrt{\frac{\hbar}{4am}}$

$\sigma_p^2 = \langle p^2\rangle - \langle p\rangle^2 = am\hbar \Rightarrow \sigma_p = \sqrt{am\hbar}$

$\sigma_x\sigma_p = \sqrt{\frac{\hbar}{4ma}}\cdot\sqrt{am\hbar} = \frac{\hbar}{2}$ // cumple! justito...

P3

$$h/\sqrt{3mk_BT} > d \Rightarrow T < \frac{h^2}{3mk_Bd^2}$$

a) electrones ($m_e = 9.1\cdot10^{-31}$ kg): $T < \frac{(6.6\cdot10^{-34})^2}{3(9.1\cdot10^{-31})(1.4\cdot10^{-23})(3\cdot10^{-10})^2} = 1,3\cdot10^5$ K

núcleos de Silicio ($m = 28m_p = 28(1,7\cdot10^{-27}) = 4,76\cdot10^{-26}$ kg

$$T < \frac{(6.6\cdot10^{-34})^2}{3(4.76\cdot10^{-26})(1.4\cdot10^{-23})(3\cdot10^{-10})^2} \approx 2,4 \text{ K}$$

b) $PV = Nk_BT$; volumen ocupado por una molécula ($N=1, V=d^3$) $\Rightarrow d = (k_BT/P)^{1/3}$

$$T < \frac{h^2}{2mk_B}\left(\frac{P}{k_BT}\right)^{2/3} \Rightarrow T^{5/2} = \frac{h^2}{2m}\frac{P^{2/3}}{k_B^{5/3}} \Rightarrow T < \frac{1}{k_B}\left(\frac{h^2}{3m}\right)^{3/5}\cdot P^{2/5}.$$

para el helio ($m = 4m_p = 6.8 \cdot 10^{-27}$ kg) a $1\,\text{atm} = 1 \cdot 10^5\ N/m^2$

$$T < \frac{1}{(1.4 \cdot 10^{-23})} \cdot \left(\frac{(6.6 \cdot 10^{-34})^2}{3(6.8 \cdot 10^{-27})}\right)^{3/5} \cdot (10^5)^{2/5} = 2.8\ K$$

para el hidrogeno ($m = 2m_p = 3.4 \cdot 10^{-27}$ kg) con $d = 10^{-2}$ m

$$T < \frac{(6.6 \cdot 10^{-34})^2}{3(3.4 \cdot 10^{-27})(1.4 \cdot 10^{-23})(10^{-2})^2} = 3.1 \cdot 10^{-14} K$$

Si el hidrogeno espacial está a 3K constante entonces es definitivamente en el regimen clásico.