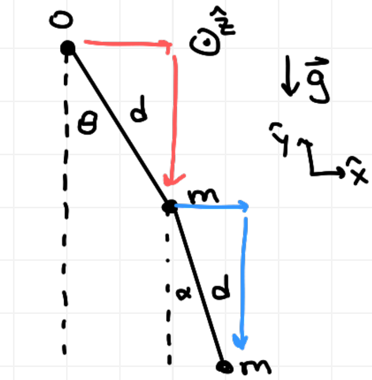
P1



A) El mom angular se define como $\vec{L} = \vec{r} \times \vec{p}$, con $\vec{p} = m\vec{v}$, así que necesitamos $\vec{r}$ para cada masa. Como nos interesa $\vec{L}_2$:

$$\vec{r}_2 = \vec{r}_1 + d\sin\alpha\,\hat{x} - d\cos\alpha\,\hat{y}$$

$$\vec{r}_1 = d\sin\theta\,\hat{x} - d\cos\theta\,\hat{y} \longrightarrow \vec{v}_1 = d\dot{\theta}(\cos\theta\,\hat{x} + \sin\theta\,\hat{y})$$

$$\vec{r}_2 = d[\hat{x}(\sin\theta + \sin\alpha) - \hat{y}(\cos\theta + \cos\alpha)] = x_2\hat{x} + y_2\hat{y}$$

$$\vec{v}_2 = d[\hat{x}(\dot{\theta}\cos\theta + \dot{\alpha}\cos\alpha) + \hat{y}(\dot{\theta}\sin\theta + \dot{\alpha}\sin\alpha)] = \dot{x}_2\hat{x} + \dot{y}_2\hat{y}$$

$$\vec{L}_2 = \vec{r}_2 \times \vec{p}_2 = m\vec{r}_2 \times \vec{v}_2 = m(x_2\dot{y}_2 - y_2\dot{x}_2)\hat{z}$$

$$\vec{r}_2 \times \vec{v}_2 = \begin{pmatrix} x_2 \\ y_2 \\ 0 \end{pmatrix} \times \begin{pmatrix} \dot{x}_2 \\ \dot{y}_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x_2\dot{y}_2 - y_2\dot{x}_2 \end{pmatrix} = (x_2\dot{y}_2 - y_2\dot{x}_2)\hat{z}$$

$$x_2\dot{y}_2 - y_2\dot{x}_2 = d(\sin\theta + \sin\alpha)\cdot d(\dot{\theta}\sin\theta + \dot{\alpha}\sin\alpha) - [-d(\cos\theta + \cos\alpha)]\cdot d(\dot{\theta}\cos\theta + \dot{\alpha}\cos\alpha)$$

$$= d^2(\dot{\theta}\sin^2\theta + \dot{\alpha}\sin^2\alpha + (\dot{\theta} + \dot{\alpha})\sin\theta\sin\alpha) + d^2(\dot{\theta}\cos^2\theta + \dot{\alpha}\cos^2\alpha + (\dot{\theta} + \dot{\alpha})\cos\theta\cos\alpha)$$

$$= d^2(\dot{\theta} + \dot{\alpha} + (\dot{\theta} + \dot{\alpha})(\sin\theta\sin\alpha + \cos\theta\cos\alpha))$$

$$= d^2(\dot{\theta} + \dot{\alpha})(1 + \cos(\theta - \alpha)) \quad \Big/ \cos(\theta - \alpha) \approx 1, \ \theta \ll 1,\ \alpha \ll 1$$

$$\approx d^2(\dot{\theta} + \dot{\alpha})(1 + 1)$$

$$\boxed{\vec{L}_2 \approx 2md^2(\dot{\theta} + \dot{\alpha})\hat{z}}$$

B) Ahora para $\vec{L}_{tot}$ necesitamos $\vec{L}_1$

$$\vec{L}_1 = m\vec{r}_1 \times \vec{v}_1$$

$$= m\, d(\sin\theta\hat{x} - \cos\theta\hat{y}) \times d\dot{\theta}(\cos\theta\hat{x} + \sin\theta\hat{y})$$

$$= md^2\dot{\theta}\left[\sin\theta\hat{x} \times (\cos\theta\hat{x} + \sin\theta\hat{y}) - \cos\theta\hat{y}(\cos\theta\hat{x} + \sin\theta\hat{y})\right]$$

$$= md^2\dot{\theta}(\sin^2\theta\,\hat{x}\times\hat{y} - \cos^2\theta\,\hat{y}\times\hat{x})$$

$$= md^2\dot{\theta}\hat{z}$$

$$\vec{L} = md^2\dot{\theta}\hat{z} + 2md^2(\dot{\theta} + \dot{\alpha})\hat{z}$$

$$\boxed{\vec{L} = md^2(3\dot{\theta} + 2\dot{\alpha})\hat{z}}$$

c) La ecuación de mov. se calcula c/r al CM del sistema, el cual es el punto central entre las masas

$$\vec{r}_{CM} = \vec{r}_1 + \frac{1}{2}d(\sin\alpha\hat{x} - \cos\alpha\hat{y})$$

$$= d\left[\hat{x}\left(\sin\theta + \frac{1}{2}\sin\alpha\right) - \hat{y}\left(\cos\theta + \frac{1}{2}\cos\alpha\right)\right]$$

Ahora, la ec. de mov. se obtiene de la variación de momento angular

$$\dot{\vec{L}} = \sum \underbrace{\vec{r}\times\vec{F}}_{\vec{\tau}\ \text{torques}}$$

Las fuerzas son las que actúan en el centro de masas, y en este caso solo sería la gravedad $\vec{F}_G = -2mg\hat{y}$

$$\vec{\tau} = \vec{r}_{CM} \times \vec{F}_G = d\left[\hat{x}\left(\sin\theta + \frac{1}{2}\sin\alpha\right) - \hat{y}\left(\cos\theta + \frac{1}{2}\cos\alpha\right)\right] \times -2mg\hat{y}$$

$$= -2mgd\left(\sin\theta + \frac{1}{2}\sin\alpha\right)\hat{z}$$

Variando el mom angular total:

$$\dot{\vec{L}} = \frac{d}{dt}\left[md^2(3\dot{\theta} + 2\dot{\alpha})\hat{z}\right] = md^2(3\ddot{\theta} + 2\ddot{\alpha})\hat{z}$$

Con lo anterior,

$$\cancel{m}d^{\cancel{2}}(3\ddot{\theta}+2\ddot{\alpha})\hat{z} = -2\cancel{m}g\cancel{d}\left(\sin\theta+\frac{1}{2}\sin\alpha\right)\hat{z}$$

$$d(3\ddot{\theta}+2\ddot{\alpha}) = -2g\left(\sin\theta+\frac{1}{2}\sin\alpha\right)$$

$$3\ddot{\theta}+2\ddot{\alpha} = -\frac{2g}{d}\left(\sin\theta+\frac{1}{2}\sin\alpha\right) \qquad /\theta,\alpha \ll 1$$

$$3\ddot{\theta}+2\ddot{\alpha} \simeq -\frac{2g}{d}\left(\theta+\frac{1}{2}\alpha\right)$$

Como $\alpha=\alpha_0\sin\omega t \longrightarrow \ddot{\alpha}=-\alpha_0\omega^2\sin\omega t$,

$$3\ddot{\theta}-2\alpha_0\omega^2\sin\omega t = -\frac{2g}{d}\left(\theta+\frac{1}{2}\alpha_0\sin\omega t\right)$$

$$\boxed{\ddot{\theta}+\frac{2g}{3d}\theta = \alpha_0\left(2\omega^2-\frac{g}{d}\right)\sin\omega t}$$

y esta es la ec. de un oscilador forzado.

D) Para que no haya comportamiento tipo osc. forzado, se debe cumplir

$$2\omega^2-\frac{g}{d}=0$$

$$2\omega^2=\frac{g}{d}$$

$$\omega^2=\frac{g}{2d}$$

$$\boxed{\omega=\sqrt{\frac{g}{2d}}}$$

E) La ec. de mov sin forzamiento queda

$$\ddot{\theta}+\underbrace{\frac{2g}{3d}}_{\Omega^2}\theta=0$$

Ω^2 —— frecuencia de oscilación

$$\theta(t)=A\sin\Omega t+B\cos\Omega t$$

$$\boxed{\theta(t)=A\sin\left(\sqrt{\frac{2g}{3d}}\,t\right)+B\cos\left(\sqrt{\frac{2g}{3d}}\,t\right)}$$