Control 3

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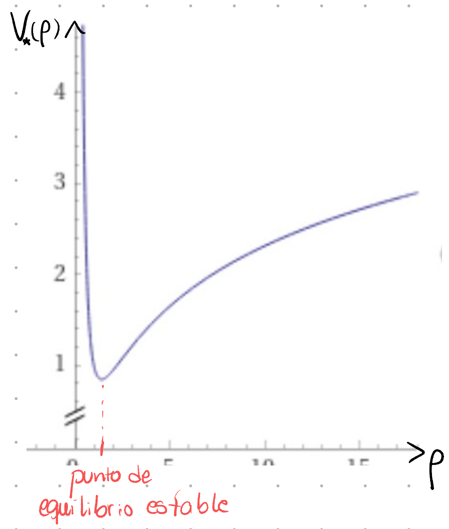
P1

a) Solo hay una fuerza central (en $\hat{\rho}$)

$$\Rightarrow m\left((\ddot{\rho}-\rho\dot{\theta}^2)\hat{\theta} + \frac{1}{\rho}\frac{d}{dt}(\rho^2\dot{\theta})\hat{\theta}\right) = -\frac{A}{\rho}\hat{\rho}$$

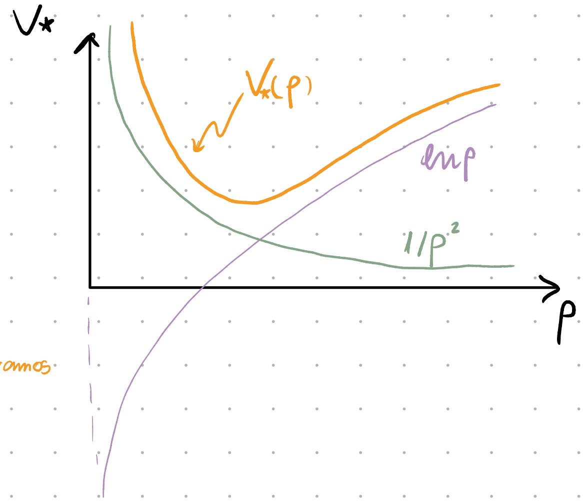
$\hat{\rho})$ $m(\ddot{\rho}-\rho\dot{\theta}^2) = -\frac{A}{\rho}$

$\hat{\theta})$ $\frac{m}{\rho}\frac{d}{dt}(\rho^2\dot{\theta}) = 0 \Rightarrow \rho^2\dot{\theta} = h_0$ constante



b) $m\left(\ddot{\rho} - \rho\frac{h_0^2}{\rho^4}\right) = -\frac{A}{\rho}$

$$\Leftrightarrow \ddot{\rho} = -\frac{A}{m}\frac{1}{\rho} + \frac{h_0^2}{\rho^3} = f(\rho)$$



c) Por enunciado $f(\rho) = -\frac{A}{m}\frac{1}{\rho} + \frac{h_0^2}{\rho^3} = -\frac{dV_*}{d\rho}$ $\quad / -\int d\rho$

$$\Rightarrow \frac{A}{m}\int\frac{d\rho}{\rho} - h_0^2\int\frac{d\rho}{\rho^3} = \int dV_*$$

$$\Leftrightarrow V_*(\rho) = \frac{A\ln(\rho)}{m} + \frac{h_0^2}{2}\frac{1}{\rho^2} + \cancel{C_1}$$ ignoramos

d) Para encontrar ρ_* usamos que en el punto de equilibrio

$$\left.\frac{dV_*}{d\rho}\right|_{\rho_*} \overset{!}{=} 0$$

$$\Leftrightarrow \frac{A}{m}\frac{1}{\rho_*} - \frac{h_0^2}{\rho_*^3} \overset{!}{=} 0 \quad /\cdot\rho_*^3$$

$$\Rightarrow \frac{A}{m}\rho_*^2 - h_0^2 = 0$$

$$\Leftrightarrow \rho_* = \pm h_0\sqrt{\frac{m}{A}} \Rightarrow \rho_* = h_0\sqrt{\frac{m}{A}}$$

Para calcular la frecuencia de pequeñas oscilaciones calculamos

$$\frac{d^2V_*}{d\rho^2} = \frac{d}{d\rho}\left(\frac{A}{m}\frac{1}{\rho} - \frac{h_0^2}{\rho^3}\right) = -\frac{A}{m}\frac{1}{\rho^2} + \frac{3h_0^2}{\rho^4}$$

$$\Rightarrow \omega_0 = \sqrt{\frac{U''(\rho_*)}{m}} = \sqrt{\left(-\frac{A}{m} \frac{A}{m} \frac{1}{h_0^2} + 3h_0^2 \frac{A^2}{m^2} \frac{1}{h_0^4} \right) \frac{1}{m}}$$

$$= \sqrt{-\frac{A^2}{m^3 h_0^2} + \frac{3A^2}{m^3 h_0^2}}$$

$$= \frac{A}{h_0} \sqrt{\frac{2}{m}}$$

P2

a) Nuestra C.I. es $r_0 = a \Rightarrow u_0 = \frac{1}{r_0} = \frac{1}{a}$ y como $\dot{r}_0 = 0$ (la velocidad es tangencial)

y $\dot{u} = -\frac{1}{r^2}\dot{r} \Rightarrow \dot{u}_0 = 0$ y como $\left.\frac{du}{dt}\right|_{t=0} = \left.\frac{du}{d\theta}\frac{d\theta}{dt}\right|_{t=0} = 0 \Rightarrow \left.\frac{du}{d\theta}\right|_{t=0} = 0$

Ocupamos Binet

$$u'' + u = -\frac{m}{L^2u^2} \cdot -m\gamma^2(4u^3 + a^2u^5)$$

$$\Leftrightarrow u'' = -u + \frac{m^2\gamma^2}{L^2}(4u + a^2u^3)$$

$$\frac{u'du'}{du} = \left(-1 + \frac{4m^2\gamma^2}{L^2}\right)u + a^2u^3 \quad / \int du$$

$$\int_{\dot{u}_0}^{\dot{u}} u'du' = \left(-1 + \frac{4m^2\gamma^2}{L^2}\right)\int_{u_0}^{u} u\,du + a^2\int_{u_0}^{u} u^3du$$

$$\frac{u'^2}{2} - \cancel{\frac{u_0'^2}{2}}^{\,0} = \left(-1 + \frac{4m^2\gamma^2}{L^2}\right)\left[\frac{u^2}{2} - \frac{u_0^2}{2}\right] + \frac{a^2}{4}\left[u^4 - u_0^4\right]$$

$$u'^2 = \left(-1 + \frac{4m^2\gamma^2}{L^2}\right)\left[u^2 - \frac{1}{a^2}\right] + \frac{a^2}{2}\left[u^4 - \frac{1}{a^4}\right]$$

Calculemos el momentum angular (que se conserva)

$$L = m\rho_0 v_{T,0} = m\cancel{a}\frac{3\gamma}{\sqrt{2}\cancel{a}} = \frac{3}{\sqrt{2}}m\gamma$$

$$\Rightarrow \frac{4m^2\gamma^2}{L^2} = 4\cancel{m^2\gamma^2}\frac{2}{9\cancel{m^2\gamma^2}} = \frac{8}{9}$$

$$\Rightarrow u' = \sqrt{\left(-1 + \frac{8}{9}\right)u^2 + \frac{a^2}{2}u^4 - \frac{1}{2a^2}}$$

$$\frac{du}{d\theta} = \sqrt{-\frac{1}{9}u^2 + \frac{a^2}{2}u^4 - \frac{1}{2a^2}}$$

$$= \sqrt{\frac{-2a^2u^2 + 9a^4u^4 - 9}{18a^2}}$$

$$\Rightarrow \int_{u_0}^{u}\frac{du}{\sqrt{-2a^2u^2 + 9a^4u^4 - 9}} = \frac{1}{3a\sqrt{2}}\int_0^{\theta} d\theta$$

hacemos el c.v. $au = \frac{1}{\cos y} \Rightarrow a\,du = \frac{\sin y}{\cos^2 y}\,dy$

$$\Rightarrow \int \frac{1}{\sqrt{-2\frac{1}{\cos^2 y} + \frac{9}{\cos^4 y} - 9}}\ \frac{1}{a}\,\frac{\sin y}{\cos^2 y}\,dy = \theta$$

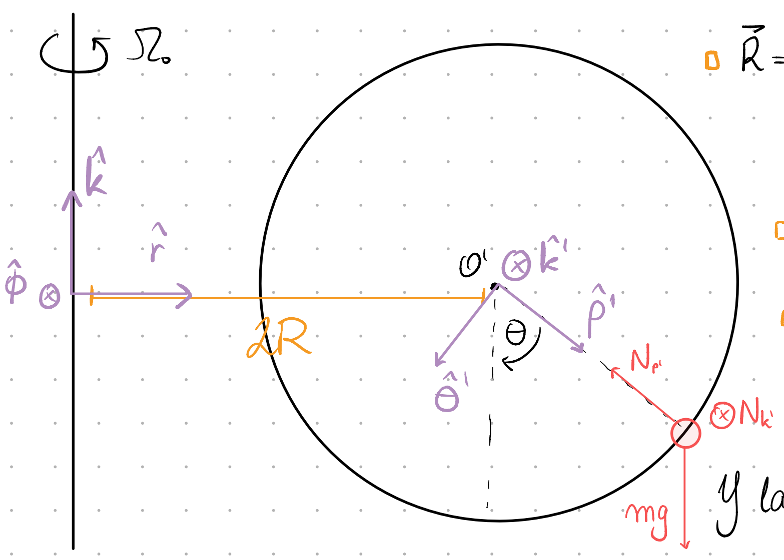
$$\Leftrightarrow \int \frac{1}{\sqrt{\frac{-2\cos^2 y + 9 - 9\cos^4 y}{\cos^4 y}}}\ \frac{1}{a}\,\frac{\sin y}{\cos^2 y}\,dy = \theta$$

$$\int \frac{\cancel{\cos^2 y}}{\sqrt{-2\cos^2 y + 9(1-\cos^4 y)}}\ \frac{1}{a}\,\frac{\sin y}{\cancel{\cos^2 y}}\,dy = \theta$$

$$\frac{1}{a}\int \frac{1}{\sqrt{-2\cos^2 y + 9\sin^4 y}}\ \sin y\,dy = \theta$$

Estamos trabajando para usted

P3



Tenemos

- $\vec{R} = 2R\hat{r} \Rightarrow \dot{\vec{R}} = 2R\dot{\phi}\hat{\phi} = 2R\Omega_0\hat{\phi}$

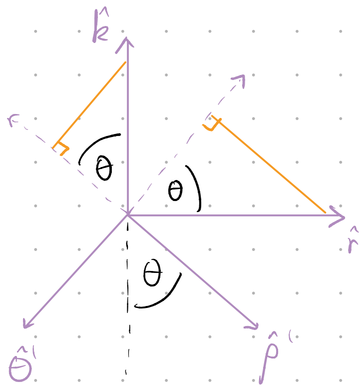
  $\Rightarrow \ddot{\vec{R}} = -2R\Omega_0^2\hat{r}$
- $\vec{\Omega} = \Omega_0\hat{k}$
- $\vec{r}\,' = R\hat{\rho}'$

  $\Rightarrow \dot{\vec{r}}\,' = R\dot{\theta}'\hat{\theta}' \Rightarrow \ddot{\vec{r}}\,' = R\ddot{\theta}'\hat{\theta}' - R\dot{\theta}'^2\hat{\rho}'$

Y las fuerzas reales son

$$\vec{F} = \sum_i \vec{F}_i = N_{\rho'}\hat{\rho}' + N_{k'}\hat{k}' - mg\hat{k}$$

Busquemos describir $(\hat{r}, \hat{\phi}, \hat{k})$ en $(\hat{\rho}', \hat{\theta}', \hat{k}')$



$$\begin{cases} \hat{r} = -\cos\theta'\hat{\theta}' + \sin\theta'\hat{\rho}' \\ \hat{k} = -\cos\theta'\hat{\rho}' - \sin\theta'\hat{\theta}' \end{cases}$$, reemplazando

- $-m\ddot{\vec{R}} = 2mR\Omega_0^2(-\cos\theta'\hat{\theta}' + \sin\theta'\hat{\rho}')$
- $\vec{\Omega} = \Omega_0(-\cos\theta'\hat{\rho}' - \sin\theta'\hat{\theta}')$
- $\vec{F} = N_{\rho'}\hat{\rho}' + N_{k'}\hat{k}' - mg(-\cos\theta'\hat{\rho}' - \sin\theta'\hat{\theta}')$

Procedamos a hacer los productos cruz

- $-m\vec{\Omega}\times(\vec{\Omega}\times\vec{r}\,') = -m\Omega_0(\cos\theta'\hat{\rho}' + \sin\theta'\hat{\theta}') \times (\Omega_0(\cos\theta'\hat{\rho}' + \sin\theta'\hat{\theta}') \times R\hat{\rho}')$

  $= -m\Omega_0^2 R(\cos\theta'\hat{\rho}' + \sin\theta'\hat{\theta}') \times (-\sin\theta'\hat{k}')$

  $= m\Omega_0^2 R\sin\theta'(-\cos\theta'\hat{\theta}' + \sin\theta'\hat{\rho}')$
- $-2m\vec{\Omega}\times\dot{\vec{r}}\,' = 2m\Omega_0(\cos\theta'\hat{\rho}' + \sin\theta'\hat{\theta}') \times R\dot{\theta}'\hat{\theta}'$

  $= 2mR\Omega_0\dot{\theta}'\cos\theta'\hat{k}'$
- $-m\dot{\vec{\Omega}}\times\vec{r}\,' = \vec{0}$, ya que $\frac{d}{dt}(\Omega_0\hat{k}) = \vec{0}$

reemplazamos en la ec. maestra

$$m(R\ddot{\theta}'\hat{\theta}' - R\dot{\theta}'^2\hat{\rho}') = N_{\rho'}\hat{\rho}' + N_{k'}\hat{k}' + mg(\cos\theta'\hat{\rho}' + \sin\theta'\hat{\theta}') + 2mR\Omega_0^2(-\cos\theta'\hat{\theta}' + \sin\theta'\hat{\rho}')$$

$$+ 2mR\Omega_0\dot{\theta}'\cos\theta'\hat{k}' + m\Omega_0^2 R\sin\theta'(-\cos\theta'\hat{\theta}' + \sin\theta'\hat{\rho}')$$

de donde obtenemos las EOM escalares

$\hat{\rho}'$) $-mR\dot{\theta}'^2 = N_{\rho'} + mg\cos\theta' + 2mR\Omega_0^2\sin\theta' + m\Omega_0^2 R\sin^2\theta'$

$\hat{\theta}'$) $mR\ddot{\theta} = mg\sin\theta' - 2mR\Omega_0^2\cos\theta' - m\Omega_0^2 R\sin\theta'\cos\theta'$

$\hat{k}'$) $0 = N_{k'} + 2mR\Omega_0\dot{\theta}'\cos\theta'$

a) De $\hat{\theta}'$) imponemos que $\ddot{\theta}(\theta=60) \overset{!}{=} 0$

$$\Rightarrow 0 \overset{!}{=} mg\frac{\sqrt{3}}{2} - 2mR\Omega_0^2\frac{1}{2} - m\Omega_0^2 R\frac{\sqrt{3}}{2}\cdot\frac{1}{2}$$

$$\Leftrightarrow \Omega_0^2\left(mR + mR\frac{\sqrt{3}}{4}\right) = mg\frac{\sqrt{3}}{2}$$

$$\Rightarrow \Omega_0 = \frac{g}{R}\frac{\sqrt{3}}{2}\left(1 + \frac{\sqrt{3}}{4}\right)^{-1}$$

b) $\hat{\theta}'$) $mR\ddot{\theta} = mg\sin\theta' - 2mR\Omega_0^2\cos\theta' - m\Omega_0^2 R\sin\theta'\cos\theta'$

c) Con $\hat{\theta}$) usamos trucazo de mecánica

$$\dot{\theta}'\frac{d\dot{\theta}'}{d\theta'} = \frac{g}{R}\sin\theta' - 2\Omega_0^2\cos\theta' - \Omega_0^2\sin\theta'\cos\theta' \quad // \int d\theta'$$

$$\int_{\dot{\theta}'=\omega_0}^{\dot{\theta}'}\dot{\theta}'d\dot{\theta}' = \frac{g}{R}\int_{60}^{\theta'}\sin\theta'd\theta' - 2\Omega_0^2\int_{60}^{\theta'}\cos\theta'd\theta' - \Omega_0^2\int_{60}^{\theta'}\sin\theta'\cos\theta'd\theta'$$

$$\frac{\dot{\theta}'^2}{2} - \frac{\omega_0^2}{2} = -\frac{g}{R}\cos\theta'\Big|_{60}^{\theta'} - 2\Omega_0^2\sin\theta'\Big|_{60}^{\theta'} + \frac{\Omega_0^2}{2}\cos^2\theta'\Big|_{60}^{\theta'}$$

$$\Leftrightarrow \dot{\theta}'(\theta') = \sqrt{\omega_0^2 - 2\frac{g}{R}\cos\theta' + \frac{g}{R} - 4\Omega_0^2\sin\theta' + 2\sqrt{3}\Omega_0^2 + \Omega_0^2\cos^2\theta' - \frac{\Omega_0^2}{4}}$$