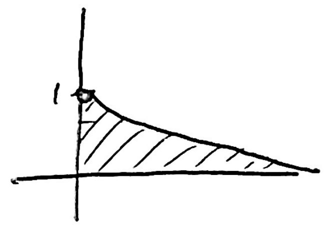
P1

a) $$1=\int_0^\infty\int_0^{e^{-x}} c\,dy\,dx=\int_0^\infty e^{-x}c\,dx=c \Rightarrow \boxed{c=1}$$

b) $x\in[0,\infty) \Rightarrow y\in\underbrace{[0,e^{-x}]}_{D_x}$



$y\in[0,1] \Rightarrow x\in\underbrace{[0,-\ln(y)]}_{D_y}$

Si $x\in[0,\infty)$

$$f_X(x)=\int_{D_x} f_{X,Y}(x,y)\,dy=\int_0^{e^{-x}}1\cdot dy=e^{-x} \Rightarrow f_X(x)=e^{-x}\mathbb{1}_{[0,\infty)}(x)$$

Si $y\in[0,1]$

$$f_Y(y)=\int_{D_y} f_{X,Y}(x,y)\,dx=\int_0^{-\ln(y)}1\,dx=-\ln(y) \Rightarrow f_Y(y)=-\ln(y)\mathbb{1}_{[0,1]}(y)$$

Como $f_X(x)\cdot f_Y(y)\neq f_{X,Y}(x,y) \Longrightarrow$ No son independientes

P2 $$f_{X,Y}(x,y)=e^{-y}\mathbb{1}_{\{0<x<y\}}$$

a) $x\in(0,\infty) \Rightarrow y\in\underbrace{(x,\infty)}_{D_x} \rightarrow f_X(x)=\left[\int_x^\infty e^{-y}dy\right]\mathbb{1}_{[0,\infty)}(x)=\boxed{e^{-x}\mathbb{1}_{[0,\infty)}(x)}$

$y\in(0,\infty) \Rightarrow x\in\underbrace{(0,y)}_{D_y} \Rightarrow f_Y(y)=\left[\int_0^y e^{-y}dx\right]\mathbb{1}_{[0,\infty)}(y)=\boxed{e^{-y}\cdot y\,\mathbb{1}_{[0,\infty)}(y)}$

Igual que en $\boxed{P1}$ No son independientes

b) $E(Y-X) = E(Y) - E(X)$

$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^{\infty} x \cdot e^{-x}\,dx = e^{-x}(-x-1)\Big|_0^{\infty} = \boxed{1}$

$E(y) = \int_{-\infty}^{\infty} y\, f_Y(y) = \int_0^{\infty} y^2 e^{-y}\,dy = e^{-y}(-y^2-2y-2)\Big|_0^{\infty} = \boxed{2}$

$\Rightarrow \boxed{E(Y-X) = 1}$

$\boxed{P3}$ $U = Y \Rightarrow \boxed{UV = X}$, $\boxed{U = Y}$, $\boxed{\text{Como } X \text{ e } Y \text{ independientes } f_{X,Y} = f_X \cdot f_Y}$

$V = \frac{X}{Y}$

$g_1(X,Y) = U = Y$, $g_2(X,Y) = V = \frac{X}{Y}$

Usando Cambio de Variable $J_g(X,Y) = \begin{bmatrix} 0 & 1 \\ \frac{1}{Y} & -\frac{X}{Y^2} \end{bmatrix}$

$\Rightarrow [D J(J_g(x,y))]^{-1} = \left[\left|\frac{-1}{y}\right|\right]^{-1} = |Y| = |U|$

$f_{U,V}(u,v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2(u,v)}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2(u,v)}{2}} \cdot |Y(u,v)|$

$= \frac{1}{2\pi} e^{-\frac{(uv)^2}{2}} \cdot e^{-\frac{u^2}{2}} \cdot |u| = \frac{|u|}{2\pi} e^{-\frac{u^2}{2}(1+v^2)}$

$$f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,du = \int_{-\infty}^{\infty} \frac{|u|}{2\pi} e^{-\frac{u^2}{2}(1+v^2)}\,du$$

$$= \int_0^\infty \frac{u}{\pi} e^{-\frac{u^2}{2}(1+v^2)}\,du \underset{2u\,du = dz}{\overset{u^2 = z}{=}} \int_0^\infty \frac{1}{\pi} e^{-\frac{z}{2}(1+v^2)} \frac{dz}{2}$$

$$= \frac{1}{2\pi} \int_0^\infty \frac{(1+v^2)}{(1+v^2)} e^{-\frac{z}{2}(1+v^2)}\,dz = \frac{1}{2\pi(1+v^2)} \int_0^\infty 2\left[\frac{(1+v^2)}{2}\right] e^{-\frac{z}{2}(1+v^2)}\,dz$$

$$= \frac{1}{\cancel{2}\pi(1+v^2)} \, -e^{-\frac{z}{2}(1+v^2)} \Big|_0^\infty = \boxed{\frac{1}{\pi(1+v^2)}}$$

P4 Me da flojera / pero No lo hagan para el examen