Pauta Proba

P1 Sea (X,Y) vector con

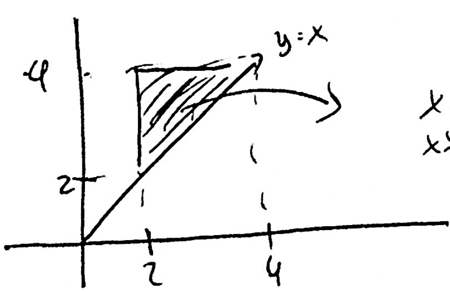
$$f_{X,Y}(x,y) = e^{-y}\,\mathbb{1}_{\{0<x<y\}}$$

a) $P(X>2 \mid Y<4)$ y $P(X>2 \mid Y=4)$

Para $$\boxed{P(X>a \mid Y<b) = \frac{P(X>a,\, Y<b)}{P(Y<b)}}$$

En este caso $\Rightarrow P(X>2 \mid Y<4) = \dfrac{P(X>2,\, Y<4)}{P(Y<4)}$

$$P(X>2,\, Y<4) = \int_2^{\infty}\int_{-\infty}^{4} e^{-y}\,\mathbb{1}_{\{0<x<y\}}\,dy\,dx$$



$x<y$
$x>2,\ y<4$

$$\Rightarrow P(X>2,\, Y<4) = \int_2^4\int_x^4 e^{-y}\,dy\,dx$$

$$= \int_2^4 -e^{-4} + e^{-x}\,dx$$

$$= e^{-2} - e^{-4} - 2e^{-4}$$

$$= \boxed{e^{-2}\left[1 - 3e^{-2}\right]}$$

$$P(Y<4) = \int_{-\infty}^{4}\int_{-\infty}^{\infty} e^{-y}\,\mathbb{1}_{\{0<x<y\}}\,dx\,dy$$

$$= \int_{0}^{4}\int_{0}^{y} e^{-y}\,dx\,dy = \int_{0}^{4} e^{-y}\cdot y\,dy = e^{-y}(-1-y)\Big|_{0}^{4}$$

$$= 1-4e^{-4}$$

$$\Rightarrow P(X>2\mid Y<4) = \frac{e^{-2}(1-3e^{-2})}{1-4e^{-4}}$$

Para $P(X>a\mid Y=b)$ hacer densidad Condicional

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \quad \begin{matrix}\to \text{Conjunta} \\ \to \text{Marginal de } Y\end{matrix}$$

Para $y>0$

$$f_Y(y) = \int_{D_Y} f_{X,Y}(x,y)\,dx \quad , \quad y\in[0,\infty), \; x\in[0,y]\in D_Y$$

$$= \int_0^y e^{-y}\,\mathbb{1}_{\{0<x<y\}}\,dx = y e^{-y}$$

$$\Rightarrow f_Y(y) = y e^{-y}\,\mathbb{1}_{\{0<y\}}$$

Para $y>0$

$$\Rightarrow f_{X|Y}(x|y) = \frac{\cancel{e^{-y}}}{y\cancel{e^{-y}}}\,\mathbb{1}_{\{0<x<y\}} = \frac{1}{y}\,\mathbb{1}_{\{0<x<y\}}$$

$$\boxed{P(X>a \mid Y=b) = \int_a^{\infty} f_{X|Y}(x \mid Y=b)\, dx}$$

$$P(X>2 \mid Y=4) = \int_2^{\infty} \frac{1}{4} \mathbb{1}_{\{0<x<4\}}\, dx = \int_2^4 \frac{1}{4} dx = \frac{2}{4} = \boxed{\frac{1}{2}}$$

b) $\boxed{E(X \mid Y=y) = \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y)\, dx}$

$$E(X \mid Y=y) = \int_{-\infty}^{\infty} \frac{x}{y} \mathbb{1}_{\{0<x<y\}}\, dx = \int_0^y \frac{x}{y} dx = \left.\frac{x^2}{2y}\right|_0^y = \boxed{\frac{y}{2}}$$

Como $E(X \mid Y=y)$ depende de $y \Rightarrow$ No son Independientes

Si $E(X) = E(X \mid Y=y) \Rightarrow$ Independientes

P2

a) $E(Z) = E(\alpha X + (1-\alpha)Y) \overset{\text{linealidad}}{=} E(\alpha X) + E((1-\alpha)Y)$

$\overset{\text{linealidad}}{=} \alpha E(X) + (1-\alpha)E(Y) \underset{E(X)=E(Y)=\mu}{=} \alpha\cdot\mu + (1-\alpha)\mu = \boxed{\mu}$

b) $Var(Z) = Var(\alpha X + (1-\alpha)Y) \overset{\text{independencia}}{=} Var(\alpha X) + Var((1-\alpha)Y)$

$\overset{\text{Prop}}{=} \alpha^2 Var(X) + (1-\alpha)^2 Var(Y)$

$= \boxed{\alpha^2\sigma_x^2 + (1-\alpha)^2\sigma_y^2}$

Para minimizar derivar respecto a $\boxed{\alpha}$

$$\frac{\partial Var(Z)}{\partial\alpha} = 2\alpha\sigma_x^2 + 2(1-\alpha)(-1)\sigma_y^2 \overset{\text{óptimo}}{=} 0$$

$$\Rightarrow \alpha\sigma_x^2 + (1-\alpha)\sigma_y^2 = 0$$

$$\Rightarrow \alpha\sigma_x^2 + \alpha\sigma_y^2 = \sigma_y^2 \Rightarrow \boxed{\alpha = \frac{\sigma_y^2}{\sigma_x^2+\sigma_y^2}}$$

$$\frac{\partial^2 Var(Z)}{\partial\alpha^2} = 2\sigma_x^2 + \sigma_y^2 > 0 \Rightarrow \text{Es mínimo}$$

$$\Rightarrow \tilde{Z} = \frac{\sigma_y^2}{\sigma_x^2+\sigma_y^2}X + \frac{\sigma_x^2}{\sigma_x^2+\sigma_y^2}Y$$ mínima varianza

P3

$$X_i = \begin{cases} 1 & \text{Si la persona } i \text{ que reserva viene a comer} \\ 0 & \text{Si la persona } i \text{ que reserva \underline{No viene} a comer} \end{cases}$$

$P(X_i = 1) = \frac{4}{5}$, $P(X_i = 0) = \frac{1}{5}$ (El 20% No viene a comer)

$\cancel{\mathbin{/\!/}}$ $S_N = \sum_{k=1}^{N} X_k$: # Personas que come

¿ $\mathbb{P}\left(\sum_{k=1}^{N} X_k \leq 80\right) \geq 0.9772$? (Es lo que piden que no coman más de 80 personas con 0.9772)

Encontrar N tq que pase esto

idea $Z = \dfrac{\frac{1}{N}\sum_{k=1}^{N} X_k - \mu}{\sigma / \sqrt{N}} \longrightarrow N(0,1)$

$$\mathbb{P}\left(\sum_{k=1}^{N} X_k \leq 80\right) \geq 0.9772 \iff \mathbb{P}\left(\frac{1}{N}\sum_{k=1}^{N} X_k \leq \frac{80}{N}\right) \geq 0.9772$$

$E(X_k) = 1 \cdot P(X_k = 1) + 0 \cdot P(X_k = 0) = \boxed{\frac{4}{5}}$

$Var(X_k) = E(X_k^2) - E(X_k)^2 = 1^2 P(X_k = 1) - \left(\frac{4}{5}\right)^2 = \frac{4}{5}\left(1 - \frac{4}{5}\right) = \boxed{\frac{4}{25}}$

$\Rightarrow \mu = \frac{4}{5} \quad , \quad \sigma = \frac{2}{5}$

$$\Rightarrow \mathbb{P}\left(\sum_{k=1}^{N} X_k \leq 80\right) \geq 0.9772 \Leftrightarrow \mathbb{P}\left(\frac{\frac{1}{N}\sum_{k=1}^{N} X_k - \frac{4}{5}}{\frac{2}{5}} \leq \frac{\frac{80}{N} - \frac{4}{5}}{\frac{2}{5}}\right) \geq 0.9772$$

$$\Leftrightarrow \mathbb{P}\left(\frac{\frac{1}{N}\sum_{k=1}^{N} X_k - \mu}{\sigma} \leq \frac{200 - 2N}{N}\right) \geq 0.9772$$

$$\Leftrightarrow \mathbb{P}\left(\frac{\frac{1}{N}\sum_{k=1}^{N} X_k - \mu}{\sigma/\sqrt{N}} \leq \frac{200 - 2N}{\sqrt{N}}\right) \geq 0.9772$$

TCL

$$\Leftrightarrow \mathbb{P}\left(Z \leq \frac{200 - 2N}{\sqrt{N}}\right) \geq 0.9772$$

Ind

$P(Z \leq 2) = 0.9772 \Rightarrow P(Z \leq h) \geq 0.9772$ si $h \geq 2$

$$\Leftrightarrow \frac{200 - 2N}{\sqrt{N}} \geq 2$$

$\Leftrightarrow 100 - N \geq \sqrt{N}$

Si $\boxed{N = 90}$ $\Rightarrow 10 \geq \sqrt{90}$ ✓

si $N = 91 \Rightarrow 9 \geq \sqrt{91}$ ✗

$\rightarrow$ ~~$(100-N)^2 \geq N^2 \Rightarrow N^2 - 200N + 100^2$~~

El máximo $\boxed{N. \text{ es } 90}$

$\boxed{P4}$ De esta pregunta la a) y b) no importan pues usaron des de Markov y Chebyshev

a) $P(X > 85) \underset{\text{Markov}}{\leq} \frac{75}{85} = \boxed{\frac{15}{19}}$

b) $P(65 < X < 85) = P(-10 < X-75 < 10) = P(|X-75| < 10)$

$= 1 - \underline{P(|X-75| \geq 10)}$

$P(|X-75| \geq 10) \underset{\text{Chebyshev}}{\leq} \frac{25}{10^2} = \boxed{\frac{1}{4}} \Rightarrow -P(|X-75| \geq 10) \geq -\frac{1}{4}$

$\Rightarrow P(65 < X < 85) = 1 - P(|X-75| \geq 10) \geq \boxed{\frac{3}{4}}$

__Obs__: Sin tener idea de que distribuye X dan calcular cotas para las probabilidades

c) Hacer algo similar a $\boxed{P3}$

Vamos a definir

X_i = Puntaje de la persona i

$\Rightarrow$ Promedio $= \frac{1}{N}\sum_{i=1}^{N} X_i$, todos los X_i son iguales

$E(X_i) = 75$, $Var(X_i) = 25$

Están pidiendo el N, para que:

$$\mathbb{P}\left(\frac{1}{N}\sum_{i=1}^{N} X_i \in [70, 80]\right) \geq 0.99$$

$$\Leftrightarrow \mathbb{P}\left(-5 < \frac{1}{N}\sum_{i=1}^{N} X_i - 75 < 5\right) \geq 0.99$$

$$\Leftrightarrow \mathbb{P}\left(-1 < \frac{\frac{1}{N}\sum_{i=1}^{N} X_i - 75}{5} < 1\right) \geq 0.99$$

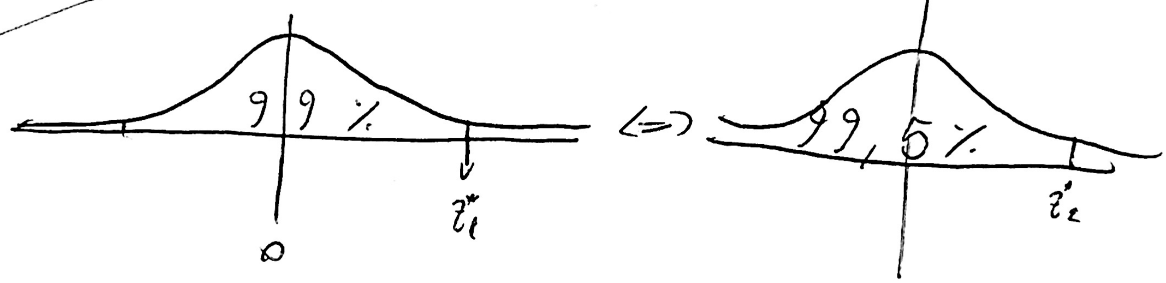
$$\Leftrightarrow \mathbb{P}\left(-\sqrt{N} < \frac{\frac{1}{N}\sum_{k=1}^{N} X_k - 75}{5/\sqrt{N}} < \sqrt{N}\right) \geq 0.99$$

$$\Leftrightarrow \mathbb{P}\left(\left|\frac{\frac{1}{N}\sum_{k=1}^{N} X_k - 75}{5/\sqrt{N}}\right| < \sqrt{N}\right) \geq 0.99$$

$$\Leftrightarrow \mathbb{P}\left(\left|\frac{\frac{1}{N}\sum_{k=1}^{N} X_k - \mu}{\sigma/\sqrt{N}}\right| < \sqrt{N}\right) \geq 0.99$$

$\iff P(|Z| \le \sqrt{N}) \ge 0.99$

$\uparrow$
TCL



(*)

$P(|Z| \le z_1^*) = 0.99$

$P(Z < z_2^*) = 0.995$

$\Rightarrow \boxed{z_1^* = z_2^*}$

Por lo que encontrar $\sqrt{N}$ usaremos que (*)

$$P(\cancel{|Z|}\, Z \le \sqrt{N}) \ge 0.995 = P(Z \le 2.57)$$

$\uparrow$ Por Tabla

$$\Rightarrow \sqrt{N} \ge 2.57$$

$$N \ge 6.6 \Rightarrow \boxed{N \ge 7}$$

Se necesita que al menos 7 alumnos den la prueba