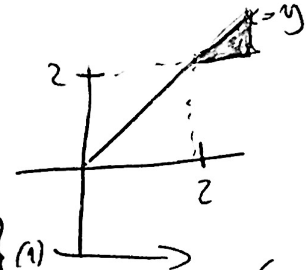
P1

$$f_{X,Y}(x,y) = e^{-(x+y)} \mathbb{1}_{\{x>0,\, y>0\}}$$

a) $\Omega_1 = \{(x,y)\in\mathbb{R}^2 : (X \geq Y \geq 2)\} =$

$= \{(x,y)\in\mathbb{R}^2 : y\in[2,\infty),\ x\in[y,\infty)\}$ (1)

$= \{(x,y)\in\mathbb{R}^2 : x\in[2,\infty),\ y\in[2,x]\}$ (2)



Cualquiera de las 2 sirven

Recuerden que para la marginal de x ($f_X(x)$) usar (2) y para la de Y usar (1)

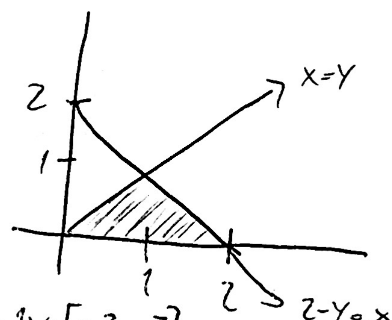
$$P(X \geq Y \geq 2) = \int_2^\infty \int_y^\infty e^{-(x+y)}\,dx\,dy$$

$$= \int_2^\infty e^{-y}\left[-e^{-x}\right]_y^\infty = \int_2^\infty e^{-2y} = \frac{e^{-2y}}{(-2)}\Big|_2^\infty = \boxed{\frac{e^{-4}}{2}}$$

$\Omega_2 = \{(x,y)\in\mathbb{R}^2 : (2-Y \geq x \geq Y)\} =$

$= \{(x,y)\in\mathbb{R}^2 : y\in[0,1],\ x\in[y, 2-y]\}$

$= \{(x,y)\in\mathbb{R}^2 : x\in[0,1],\ y\in[0,x]\} \cup \{\ '' : x\in[1,2],\ y\in[0,2-x]\}$



Note la 2da opción es una unión por lo que serían 2 integrales sumadas

$$P(Y-2 \geq X \geq Y) = \int_0^1 \int_y^{2-y} e^{-(x+y)}\,dx\,dy = \int_0^1 e^{-y}\left[-e^{-x}\right]_y^{2-y} dy$$

$$= \int_0^1 e^{-y}\left[-e^{-2+y} + e^{-y}\right] dy = \int_0^1 -e^{-2}dy + \int_0^1 e^{-2y}dy$$

$$= -e^{-2} + \left(\frac{-e^{-2y}}{2}\right)\Big|_0^1 = \frac{-3e^{-2}}{2} + \frac{1}{2} = \boxed{\frac{1-3e^{-2}}{2}}$$

b) $E(X) = \int_0^\infty \int_0^\infty x \, f_{X,Y}(x,y)\, dy\, dx$ (También se puede primero calcular la marginal $f_X(x)$ y luego $\int_{-\infty}^{\infty} x f_X(x)\,dx$)

$$= \int_0^\infty e^{-x} x \int_0^\infty e^{-y}\, dy\, dx = \int_0^\infty x e^{-x}\, dx = \left[-xe^{-x} - e^{-x}\right]\Big|_0^\infty = 1$$

$$E(y) = \int_0^\infty e^{-y} \cdot y \int_0^\infty x\, dx\, dy \underset{\text{Análogo}}{=} 1$$

$$E(X+Y) = E(X) + E(Y) = 2$$

$E(XY) \overset{\text{opción 1}}{=}$ Probar que son independientes $\Rightarrow E(XY) = E(X)\cdot E(Y)$

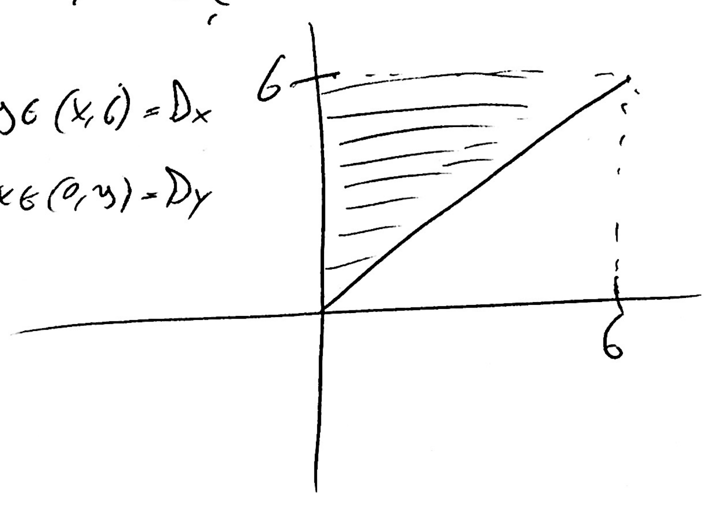
$$\overset{\text{opción 2}}{=} \int_0^\infty \int_0^\infty xy\, e^{-(x+y)}\, dx\, dy$$

$$= \int_0^\infty y e^{-y} \left[\int_0^\infty x e^{-x}\, dx\right] dy = \int_0^\infty y e^{-y} \cdot 1\, dy = 1$$

$\boxed{P2}$ ¿Cuándo $0 < x < y < 6$?

$x \in (0,6) \Rightarrow y \in (x,6) = D_x$

$y \in (0,6) \Rightarrow x \in (0,y) = D_y$



a) $\underline{\text{Para } x\in(0,6)}$

$$f_X(x) = \int_{D_X} f_{X,Y}(x,y)\,dy = \int_x^6 \frac{x}{36}\,dy = \frac{(6-x)x}{36}$$

$$\boxed{f_X(x) = \frac{(6-x)x}{36}\cdot \mathbb{1}_{(0,6)}(x)}$$

$\underline{\text{Para } y\in(0,6)}$

$$f_Y(y) = \int_{D_Y} f_{X,Y}(x,y)\,dx = \int_0^y \frac{x}{36}\,dx = \frac{x^2}{72}\Big|_0^y = \frac{y^2}{72}$$

$$\Rightarrow \boxed{f_Y(y) = \frac{y^2}{72}\cdot \mathbb{1}_{(0,6)}(y)}$$

Observando $f_X(x)\cdot f_Y(y) \neq f_{X,Y}(x,y)$ No son independientes

b) $G = Y - X \Rightarrow E(G) = E(Y) - E(X)$

$$E(Y) = \int_0^6 y\cdot\frac{y^2}{72}\,dy = \int_0^6 \frac{y^3}{72} = \frac{y^4}{4\cdot 72}\Big|_0^6 = \frac{\cancel{36}\cdot\cancel{36}^{\,9}}{\cancel{4}\cdot 2\cdot\cancel{36}} = \boxed{\frac{9}{2}}$$

$$E(X) = \int_0^6 x\cdot\frac{(6-x)x}{36}\,dx = \int_0^6 \frac{\cancel{6}x^2}{6} - \frac{x^3}{36}\,dx = \frac{x^3}{18} - \frac{x^4}{4\cdot 36}\Big|_0^6$$

$$= \frac{6\cdot\cancel{36}^{\,2}}{\cancel{18}} - \frac{\cancel{36}^{\,9}\cdot\cancel{36}}{\cancel{4}\cdot\cancel{36}} = 12 - 9 = 3$$

$$\Rightarrow E(G) = \frac{9}{2} - 3 = \boxed{\frac{3}{2}}$$

$\boxed{P3}$

$$f_{X,Y}(x,y) = \begin{cases} \lambda^2 e^{-\lambda(x+y)} & (x,y) \in [0,\infty)^2 \\ 0 & \sim \end{cases}$$

Sea $Z = \dfrac{X}{X+Y}$, $W = X+Y$ Calcule las marginales Z, W

$$\left.\begin{aligned} g_1(X,Y) &= Z = \frac{X}{X+Y} \\ g_2(X,Y) &= W = X+Y \end{aligned}\right\} \boxed{\begin{aligned} Z\cdot W &= X \\ W - ZW &= Y \end{aligned}}$$

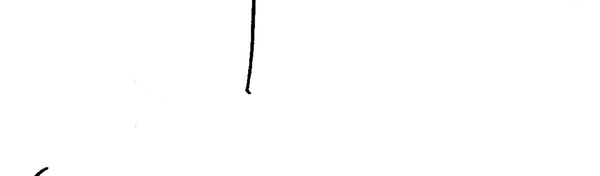
$$J_g = \begin{bmatrix} \dfrac{Y}{(X+Y)^2} & \dfrac{-X}{(X+Y)^2} \\ 1 & 1 \end{bmatrix} \Rightarrow \mathrm{Det}(J_g) = \frac{1}{X+Y}$$

$$\Rightarrow |\mathrm{Det}(J_g)|^{-1} = |X+Y| = |W|$$

Usando $f_{Z,W}(z,w) = f_{X,Y}(x,y) \cdot |\mathrm{Det}(J_g)|^{-1}$

$$f_{Z,W}(z,w) = \lambda^2 e^{-\lambda w} \cdot |w| \cdot \underbrace{\mathbb{1}_{\{0 < zw < \infty\}} \cdot \mathbb{1}_{\{0 \le w(1-z) < \infty\}}}_{\circledast}$$

$\circledast =$



$$= \left.\begin{aligned} z &\in [0,1] \\ w &\in [0,\infty) \end{aligned}\right\} \Rightarrow w \ge 0$$

$$\Rightarrow \quad f_{Z,W}(z,w) = \lambda^2 e^{-w} \cdot w \; \mathbb{1}_{\{0 \le w < \infty\}} \cdot \mathbb{1}_{\{0 \le z \le 1\}}$$

$$f_w(w) = \int_0^1 f_{w,z}(w,z)\,dz = \int_0^1 \lambda^2 e^{-\lambda w}\cdot w\,dz = \boxed{\lambda^2 e^{-\lambda w}\cdot w\,\mathbb{1}_{\{0\le w<\infty\}}}$$

$$f_z(z) = \int_0^\infty \lambda^2 e^{-\lambda w}\cdot w\cdot \mathbb{1}_{\{0\le z\le 1\}}\,dw = \lambda^2\,\mathbb{1}_{\{0\le z\le 1\}}\left[-\frac{e^{-\lambda w}}{\lambda}\cdot w + \frac{e^{-\lambda w}}{\lambda^2}\right]_0^\infty$$

$$= \cancel{\frac{\lambda^2}{\lambda^2}}\,\boxed{\mathbb{1}_{\{0\le z\le 1\}}}$$, Si son independientes y $Z \sim \text{Uniforme}(0,1)$

$$\Rightarrow \text{Var}(Z) = \frac{(b-a)^2}{12} = \boxed{\frac{1}{12}}$$

P4

$\theta \sim \text{Uniforme}(0, \frac{\pi}{4})$

$$\Rightarrow Y = \tan(\theta) = g(\theta)$$

$$F_Y(y) = P(Y \le y) = P(\tan(\theta) \le y)$$
$$= P(\theta \le \text{Arctan}(y))$$

Caso 1 $\quad \text{Arctan}(y) < 0 \Rightarrow P(\theta \le \text{Arctan}(y)) = 0$

Caso 2 $\quad \text{Arctan}(y) \in (0, \frac{\pi}{4}) \Rightarrow P(\theta \le \text{Arctan}(y)) = \int_0^{\text{Arctan}(y)} \frac{1}{\frac{\pi}{4}}\,d\theta = \frac{\text{Arctan}(y) - 0}{\frac{\pi}{4}} = \frac{4}{\pi}\text{Arctan}(y)$

Caso 3 $\quad \text{Arctan}(y) \ge \frac{\pi}{4} \Rightarrow P(\theta \le \text{Arctan}(y)) \ge P(\theta \le \frac{\pi}{4}) = 1$

$$\Rightarrow F_Y(y) = \begin{cases} 0 & y < 0 \Leftrightarrow \text{Arctan}(y) < 0 \\ \frac{4}{\pi}\text{Arctan}(y) & y \in (0,1) \Leftrightarrow \text{Arctan}(y) \in (0, \frac{\pi}{4}) \\ 1 & y \in [1, \infty) \Leftrightarrow \text{Arctan}(y) \in [\frac{\pi}{4}, \frac{\pi}{2}) \end{cases}$$

$$\Rightarrow f_Y(y) = \begin{cases} 0 & \\ \frac{4}{\pi}\frac{1}{1+y^2} & \text{si } y \in (0,1) \\ 0 & \end{cases}$$

$$\Rightarrow f_Y(y) = \frac{4}{\pi}\cdot\frac{1}{1+y^2}\mathbb{1}\{0 \leq y \leq 1\}$$

$$\Rightarrow E(Y) = \int_0^1 \frac{4}{\pi}\left(\frac{1}{2}\cdot\frac{2}{1}\right)\frac{y}{1+y^2} = \frac{4}{\pi}\cdot\frac{1}{2}\cdot\ln(1+y^2)\Big|_0^1 = \boxed{\frac{2}{\pi}\ln(2)}$$

Propuesto 1

$X \in \mathbb{N}$ e $y \in \{x, x+1, \ldots\}$ { Marginal X

$\Leftrightarrow y \in \mathbb{N}$ e $X \in \{0, \ldots, y\}$ { Marginal y

$$1 = \sum_{y\in\mathbb{N},\, x\in\{0\ldots y\}} p(x,y) = e^{-2}\sum_{y\in\mathbb{N},\, x\in\{0\ldots y\}} \frac{1}{x!\cdot(y-x)!}\cdot\frac{y!}{y!} = e^{-2}\sum_{y=0}^{\infty}\frac{1}{y!}\cdot\underbrace{\sum_{x=0}^{y}\binom{y}{x}}_{2^y}$$

$$= e^{-2}\sum_{y=0}^{\infty}\frac{2^y}{y!} = e^{-2}\cdot e^{2} = \boxed{1}$$

$$P_X(x) = \sum_{y=x}^{\infty}\frac{e^{-2}}{x!\cdot(y-x)!} = \frac{e^{-2}}{x!}\cdot\sum_{y=0}^{\infty}\frac{1}{y!} = \frac{e^{-2}}{x!}\cdot e^{1} = \boxed{\frac{e^{-1}}{x!}, \quad x \in \{0,\ldots,\infty}$$

$$P_Y(y) = \sum_{x=0}^{y}\frac{e^{-2}}{x!(y-x)!}\cdot\frac{y!}{y!} = \frac{e^{-2}}{y!}\cdot\sum_{x=0}^{y}\binom{y}{x} = \boxed{\frac{e^{-2}\cdot 2^y}{y!}, \quad y \in \{0,\ldots\infty}$$

X no es independiente de Y

b) $Z = Y - X$, Sea ~~$z \in \mathbb{N}$~~ $z \in \mathbb{N}$, $x \in \mathbb{N}$

$$P(Z = z, X = x) = P(Y - X = z, X = x)$$
$$= P(Y = z + x, X = x)$$
$$= \boxed{\frac{e^{-2}}{x! \cdot z!}}$$

$$P(Z = z) = \sum_{x \in \mathbb{N}} P(Z = z, X = x) = \sum_{x \in \mathbb{N}} \frac{e^{-2}}{x! \cdot z!} = \boxed{\frac{e^{-1}}{z!}}$$

$$P(X = x) = \frac{e^{-1}}{x!} \Rightarrow \boxed{P(X = x) \cdot P(Z = z) = P(Z = z, X = x)}$$

Son independientes

Propuesto 2 olvidar ☺