Pauta T3

P1) a) Notamos que $\{Y_j = 1\} = \{j \in A\}$

Como el cjto A se saca al azar (uniforme), basta analizar la prob como casos fav./casos tot.

Casos fav: Basta fijar que $j \in A$, el resto de bolas $(k-1)$ pueden ser cualquiera $\Rightarrow |\{j \in A\}| = \binom{n-1}{k-1}$

Casos tot: Cualq. subcjto de tamaño k: $|Tot| = \binom{n}{k}$

$$\Rightarrow \mathbb{P}(Y_j = 1) = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{\frac{(n-1)!}{(k-1)!\,(n-k)!}}{\frac{n!}{k!\,(n-k)!}} = \frac{(n-1)!\,k!}{(k-1)!\,n!} = \frac{k}{n}$$

Luego $\mathbb{E}(Y_j) = 1 \cdot \mathbb{P}(Y_j = 1) + 0 \cdot \mathbb{P}(Y_j = 0) = k/n$

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b) Por linealidad de la esperanza:

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_{j=1}^{r} Y_j\right) = \sum_{j=1}^{r} \mathbb{E}(Y_j) = \sum_{j=1}^{r} k/n = \frac{rk}{n}$$

P2) Como $X = R\cos(\Theta)$; $Y = R\,\text{sen}(\Theta)$

$\Rightarrow R = \sqrt{X^2+Y^2}$; $\Theta = \text{arctg}(Y/X)$

$\therefore (R,\Theta) = h(X,Y)$, con h inyectiva por ser cambio variable polares

Por TCV:

$$f_{R,\Theta}(r,\theta) = |\det(J_h(h^{-1}(r,\theta)))|^{-1} f_{X,Y}(h^{-1}(r,\theta)) = |\det(J_{h^{-1}}(r,\theta))|\, f_{X,Y}(h^{-1}(r,\theta))$$

↳ Del enunciado $h^{-1}(r,\theta) = (r\cos\theta, r\,\text{sen}\,\theta)$

↳ $J_{h^{-1}}(r,\theta) = \begin{pmatrix} \cos\theta & -r\,\text{sen}\,\theta \\ \text{sen}\,\theta & r\cos\theta \end{pmatrix} \Rightarrow \det(J_{h^{-1}}) = r$

$$\therefore f_{R,\Theta}(r,\theta) = \left(\frac{1}{r}\right)^{-1} f_X(r\cos\theta) f_Y(r\,\text{sen}\,\theta) = \left(\frac{1}{r}\right)^{-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2\cos^2\theta}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2\text{sen}^2\theta}{2}}$$

$$= r\,\frac{1}{2\pi}\, e^{-\frac{r^2}{2}}$$

- Notemos que $R = \sqrt{X^2+Y^2} \Rightarrow R \geq 0$

  $\Theta = \text{arctg}(Y/X) \Rightarrow \Theta \in (-\pi,\pi)$

$$\Rightarrow f_{R,\Theta}(r,\theta) = r e^{-r^2/2} \mathbb{1}_{\{r>0\}} \cdot \frac{1}{2\pi} \mathbb{1}_{\{-\pi<\theta<\pi\}}$$

$$= f_R(r)\cdot f_\Theta(\theta)$$

Esto nos dice que R y Θ son indep. con R tq $f_R(r) = r e^{-r^2/2} \mathbb{1}_{\{r>0\}}$

Θ tq $f_\Theta(\theta) = \frac{1}{2\pi} \mathbb{1}_{\{-\pi<\theta<\pi\}}$

ie, $\Theta \sim \text{Unif}(-\pi,\pi)$ ∎

P3) a) $f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy = \int_{-\infty}^{+\infty} \frac{6}{7}\left(x^2 + \frac{x\cdot y}{2}\right) \mathbb{1}_{\{x\in(0,1)\}} \mathbb{1}_{\{y\in(0,2)\}}\,dy$

$= \int_0^2 \frac{6}{7} x^2 \mathbb{1}_{\{x\in(0,1)\}} + \frac{6}{7}\frac{xy}{2} \mathbb{1}_{\{x\in(0,1)\}}\,dy$

$= \mathbb{1}_{\{x\in(0,1)\}} \left(\int_0^2 \frac{6}{7} x^2\,dy + \int_0^2 \frac{6}{7}\frac{xy}{2}\,dy \right)$

$= \mathbb{1}_{\{x\in(0,1)\}} \frac{6}{7} x \left(x\int_0^2 dy + \frac{1}{2}\int_0^2 y\,dy \right)$

$= \frac{6}{7} x \, \mathbb{1}_{\{x\in(0,1)\}} \left(2x + \frac{1}{2}\cdot 2 \right)$

$= \frac{6}{7} x (2x+1) \mathbb{1}_{\{x\in(0,1)\}}$

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b) Sea $x \in (0,1)$:

$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\frac{6}{7}\left(x^2 + \frac{xy}{2}\right)\mathbb{1}_{\{y\in(0,2)\}}}{\frac{6}{7}x(2x+1)}$

$= \frac{\cancel{\frac{6}{7}x}\left(x + \frac{y}{2}\right)\mathbb{1}_{\{y\in(0,2)\}}}{\cancel{\frac{6}{7}x}(2x+1)} = \frac{x + \frac{y}{2}}{2x+1}\mathbb{1}_{\{y\in(0,2)\}}$

$\therefore\ \mathbb{P}(Y<X) = \int_{-\infty}^{+\infty} \mathbb{P}(Y<x \mid X=x) f_X(x)\,dx = \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{x} f_{Y|X}(y|x)\,dy\right) f_X(x)\,dx$

$= \int_{-\infty}^{+\infty}\int_{-\infty}^{x} f_{X,Y}(x,y)\,dy\,dx = \int_{-\infty}^{+\infty}\int_{-\infty}^{x} \frac{6}{7}x\left(x+\frac{y}{2}\right)\mathbb{1}_{\{x\in(0,1)\}}\mathbb{1}_{\{y\in(0,2)\}}\,dy\,dx$

$= \int_0^1 \frac{6}{7}x\left(\int_{-\infty}^{x}\left(x+\frac{y}{2}\right)\mathbb{1}_{\{y\in(0,2)\}}\,dy\right)dx = \int_0^1 \frac{6}{7}x\left(\int_0^{x}\left(x+\frac{y}{2}\right)\mathbb{1}_{\{y\in(0,2)\}}\,dy\right)dx$

P3) b) $\mathbb{P}(Y<X) = \int_0^1 \frac{6}{7}x \int_0^x \left(x+\frac{y}{2}\right) \mathbb{1}_{\{y\in(0,2)\}}\,dy\,dx$

$$= \int_0^1 \frac{6}{7}x\left(x\int_0^x dy + \frac{1}{2}\int_0^x y\,dy\right)dx$$

$$= \int_0^1 \frac{6}{7}x\left(x^2 + \frac{1}{2}\cdot\frac{x^2}{2}\right)dx = \int_0^1 \frac{6}{7}x\cdot\frac{5}{4}x^2\,dx$$

$$= \int_0^1 \frac{30}{28}x^3\,dx = \frac{15}{14}\int_0^1 x^3\,dx = \frac{15}{14}\left(\frac{x^4}{4}\right)_0^1 = \frac{15}{14}\cdot\frac{1}{4}$$

$$\Rightarrow \mathbb{P}(Y<X) = \frac{15}{56} \quad \blacksquare$$

P3) c) $f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx = \int_{-\infty}^{+\infty} \frac{6}{7}x\left(x+\frac{y}{2}\right)\mathbb{1}_{\{x\in(0,1)\}}\mathbb{1}_{\{y\in(0,2)\}}\,dx$

$$= \mathbb{1}_{\{y\in(0,2)\}}\left(\int_0^1 \frac{6}{7}x^2\,dx + \int_0^1 \frac{6}{7}\frac{xy}{2}\,dx\right)$$

$$= \mathbb{1}_{\{y\in(0,2)\}}\frac{6}{7}\left(\int_0^1 x^2\,dx + \frac{y}{2}\int_0^1 x\,dx\right)$$

$$= \mathbb{1}_{\{y\in(0,2)\}}\frac{6}{7}\left(\frac{1}{3}+\frac{y}{4}\right) = \left(\frac{6}{21}+\frac{6}{28}y\right)\mathbb{1}_{\{y\in(0,2)\}}$$

$\Rightarrow$ • $\mathbb{E}(Y) = \int_{-\infty}^{+\infty} y\,f_Y(y)\,dy = \int_0^2 y\left(\frac{6}{21}+\frac{6}{28}y\right)dy = \frac{6}{21}\int_0^2 y\,dy + \frac{6}{28}\int_0^2 y^2\,dy$

$$= \frac{6}{21}\cdot 2 + \frac{6}{28}\cdot\frac{8}{3} = \frac{12}{21}+\frac{4}{7} = \frac{24}{21} = \frac{8}{7}$$

• $\mathbb{E}(Y^2) = \int_{-\infty}^{+\infty} y^2 f_Y(y)\,dy = \frac{6}{21}\int_0^2 y^2\,dy + \frac{6}{28}\int_0^2 y^3\,dy$

$$= \frac{6}{21}\cdot\frac{8}{3} + \frac{6}{28}\cdot 4 = \frac{4}{7}+\frac{6}{7} = \frac{10}{7}$$

$\therefore$ $\mathrm{Var}(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 = \frac{10}{7} - \left(\frac{8}{7}\right)^2 = \frac{70}{49} - \frac{64}{49} = \frac{6}{49} \quad \blacksquare$