C3 P y E

P1) a) Tenemos $(X, Z) = h(X, Y) = \left(X, \frac{X}{X+Y}\right)$

- h es inyectiva:

Sean $x_1, y_1, x_2, y_2 \in \mathbb{R}$: $h(x_1, y_1) = h(x_2, y_2)$

$$\Rightarrow \left(x_1, \frac{x_1}{x_1+y_1}\right) = \left(x_2, \frac{x_2}{x_2+y_2}\right)$$

$$\Rightarrow x_1 = x_2 \;/\!/ \quad \wedge \quad \frac{x_1}{x_1+y_1} = \frac{x_2}{x_2+y_2}$$

$$\Rightarrow \frac{1}{x_1+y_1} = \frac{1}{x_2+y_2}$$

$$\Rightarrow x_1 + y_1 = x_2 + y_2 \Rightarrow y_1 = y_2 \;/\!/$$

- Por TCV:

$$f_{X,Z}(x,z) = \left|\det(J_h(h^{-1}(x,z)))\right|^{-1} f_{X,Y}(h^{-1}(x,z)) = \left|\det(J_{h^{-1}}(x,z))\right| f_{X,Y}(h^{-1}(x,z))$$

* $h(x,y) = \left(x, \frac{x}{x+y}\right) \Rightarrow h^{-1}(x,z) = \left(x, \frac{x}{z} - x\right)$

* $J_{h^{-1}}(x,z) = \begin{pmatrix} 1 & \frac{1}{z} - 1 \\ 0 & -\frac{x}{z^2} \end{pmatrix} \Rightarrow \det(J_{h^{-1}}(x,z)) = -\frac{x}{z^2}$

* X es el valor de una exp, Z es división de valores $\geq 0 \Rightarrow x, z \geq 0$

$$\therefore f_{X,Z}(x,z) = \left|-\frac{x}{z^2}\right| f_X(x) f_Y\left(\frac{x}{z} - x\right) = \frac{x}{z^2} \cdot \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda\left(\frac{x}{z} - x\right)} \mathbb{1}_{x>0} \mathbb{1}_{\frac{x}{z} - x > 0}$$

$$= \frac{x}{z^2} \lambda e^{-\lambda x} \lambda e^{-\lambda \frac{x}{z}} e^{\lambda x} \mathbb{1}_{x>0} \mathbb{1}_{\frac{x}{z} > x} = \frac{x}{z^2} \lambda^2 e^{-\lambda \frac{x}{z}} \mathbb{1}_{x>0} \mathbb{1}_{0<z<1}$$

* Usamos que X, Y indep

* Notemos $\frac{x}{z} - x > 0 \Rightarrow \frac{x}{z} > \cancel{x}$. Como $x > 0 \Rightarrow z > 0$

Luego $\frac{1}{z} > 1 \Rightarrow z < 1 \quad \therefore 0 < z < 1$

P1) b) Calculemos $f_Z(z)$:

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{X,Z}(x,z)\,dx = \int_{-\infty}^{+\infty} \lambda^2 \frac{x}{z^2} e^{-\lambda \frac{x}{z}} \mathbb{1}_{x>0} \mathbb{1}_{0<z<1}\,dx$$

$$= \frac{\lambda^2}{z} \mathbb{1}_{0<z<1} \int_0^{+\infty} \frac{x}{z} e^{-\lambda \frac{x}{z}}\,dx$$

Def $u = \frac{x}{z} \Rightarrow du = \frac{1}{z}dx$

$$\therefore f_Z(z) = \frac{\lambda^2}{z} \mathbb{1}_{0<z<1} \int_0^{+\infty} u e^{-\lambda u} z\,du = \lambda^2 \mathbb{1}_{0<z<1} \int_0^{+\infty} u e^{-\lambda u}\,du$$

$$= \lambda \mathbb{1}_{0<z<1} \underbrace{\int_0^{+\infty} u \lambda e^{-\lambda u}\,du}_{\mathbb{E}(U),\ \text{con } U \sim Exp(\lambda)} = \lambda \mathbb{1}_{0<z<1} \cdot 1/\lambda = \mathbb{1}_{0<z<1}$$

$\therefore f_Z(z) = \mathbb{1}_{0<z<1} \Rightarrow Z \sim Unif(0,1)$ ∎

P2) a) $\mathbb{P}(\forall i=1..n : U_i \in (s,t]) \overset{\text{indep}}{=} \prod_{i=1}^{n} \mathbb{P}(U_i \in (s,t]) = \prod_{i=1}^{n} [F_{U_i}(t) - F_{U_i}(s)]$

$$= \prod_{i=1}^{n} [t-s] = (t-s)^n \quad \blacksquare$$

b) Notemos que el evento:

- $\{S>s\} \Leftrightarrow \{\min(U_1 .. U_n) > s\} \Leftrightarrow \{\forall i=1..n : U_i > s\}$
- $\{T\le t\} \Leftrightarrow \{\max(U_1 .. U_n) \le t\} \Leftrightarrow \{\forall i=1..n : U_i \le t\}$

Luego $\mathbb{P}(S>s, T\le t) = \mathbb{P}(\forall i=1..n : U_i > s, U_i \le t) = \mathbb{P}(\forall i=1..n : U_i \in (s,t]) = (t-s)^n \quad \blacksquare$

c) Dado $0 \le s < t \le 1$ fijos

Por prob totales: $\mathbb{P}(T\le t) = \mathbb{P}(T\le t, S>s) + \mathbb{P}(T\le t, S\le s)$

$$\Rightarrow \mathbb{P}(T\le t, S\le s) = \mathbb{P}(T\le t) - \mathbb{P}(T\le t, S>s) \,//$$

Luego $\mathbb{P}(T\le t) = \mathbb{P}(\forall i=1..n : U_i \le t) = \prod_{i=1}^{n} \mathbb{P}(U_i \le t) = \prod_{i=1}^{n} F_{U_i}(t) = \prod_{i=1}^{n} t = t^n$

$\therefore \mathbb{P}(T\le t, S\le s) = t^n - (t-s)^n$

- Por lo tanto $F_{S,T}(s,t) = t^n - (t-s)^n$

Por def $f_{S,T}(s,t) = \dfrac{\partial^2 F_{S,T}}{\partial s \, \partial t}(s,t) = \dfrac{\partial}{\partial t}\left(\dfrac{\partial F_{S,T}}{\partial s}\right)(s,t)$

$$= \frac{\partial}{\partial t}\left(n(t-s)^{n-1}\right) = n(n-1)(t-s)^{n-2}$$

$\therefore$ Para $0 \le s < t \le 1$

$$f_{S,T}(s,t) = n(n-1)(t-s)^{n-2} \quad \blacksquare$$

P3) a) $\mathbb{E}(e^{tX}) = \int_{-\infty}^{+\infty} e^{tx} f_X(x)\,dx = \int_{-\infty}^{+\infty} e^{tx} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2} + tx} dx$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-(x-\mu)^2 + 2\sigma^2 tx}{2\sigma^2}} dx$$

- Trabajemos el término dentro de la exponencial:

$$2\sigma^2 tx - (x-\mu)^2 = 2\sigma^2 tx - x^2 + 2x\mu - \mu^2 = -x^2 + 2(\sigma^2 t + \mu)x - \mu^2$$

$$= -x^2 + 2(\sigma^2 t + \mu)x - \mu^2 + (\sigma^2 t + \mu)^2 - (\sigma^2 t + \mu)^2$$

$$= -\left[x^2 - 2(\sigma^2 t + \mu)x + (\sigma^2 t + \mu)^2\right] + (\sigma^2 t + \mu)^2 - \mu^2$$

$$= -\left[x - (\sigma^2 t + \mu)\right]^2 + (\sigma^2 t + \mu)^2 - \mu^2$$

$\therefore$ $\mathbb{E}(e^{tX}) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-[x-(\sigma^2 t + \mu)]^2 + (\sigma^2 t + \mu)^2 - \mu^2}{2\sigma^2}} dx$

$$= e^{\frac{(\sigma^2 t + \mu)^2 - \mu^2}{2\sigma^2}} \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\frac{(x-(\sigma^2 t + \mu))^2}{\sigma^2}} dx}_{=1,\text{ pues es igual a } \int_{-\infty}^{+\infty} f_Y(y)\,dy,\text{ con } Y \sim N(\sigma^2 t + \mu,\, \sigma^2)}$$

$\Rightarrow \mathbb{E}(e^{tX}) = e^{\frac{(\sigma^2 t + \mu)^2 - \mu^2}{2\sigma^2}} = e^{\frac{\sigma^4 t^2 + 2\sigma^2 t\mu + \mu^2 - \mu^2}{2\sigma^2}} = e^{\frac{\sigma^4 t^2 + 2\sigma^2 t\mu}{2\sigma^2}} = e^{\frac{\sigma^2 t^2 + 2t\mu}{2}}$

$$= e^{t\left(\frac{\sigma^2 t}{2} + \mu\right)}$$

$\blacksquare$

P3) b) Usando la parte (a):

- $\mathbb{E}(Y) = \mathbb{E}(e^X) = e^{\frac{\sigma^2}{2}+\mu}$ ▨

- $Var(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2$

Calculemos $\mathbb{E}(Y^2)$:

$$\mathbb{E}(Y^2) = \mathbb{E}((e^X)^2) = \mathbb{E}(e^{2X}) = e^{2(\sigma^2+\mu)}$$ //

$$\Rightarrow Var(Y) = e^{2(\sigma^2+\mu)} - \left(e^{\left(\frac{\sigma^2}{2}+\mu\right)}\right)^2 = e^{2(\sigma^2+\mu)} - e^{2\left(\frac{\sigma^2}{2}+\mu\right)} = e^{2\sigma^2+2\mu} - e^{\sigma^2+2\mu}$$

$$= e^{2\sigma^2} \cdot e^{2\mu} - e^{\sigma^2} \cdot e^{2\mu} = e^{2\mu}\left(e^{2\sigma^2} - e^{\sigma^2}\right)$$ ▨