Aux 9 PyE)

P1) a) Notemos que $X^2 \sim$ Bernoulli (p), luego:

$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = p - p^2 = p(1-p)$$

P1) b) $$\mathbb{E}(X^2) = \sum_{k=0}^{n} k^2\, \mathbb{P}(X=k) = \sum_{k=1}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k^2 \frac{n!}{k!\,(n-k)!}\, p^k (1-p)^{n-k} = \sum_{k=1}^{n} k \frac{n!}{(n-k)!\,(k-1)!}\, p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n-1} (k+1) \frac{n!}{(n-1-k)!\,k!}\, p^{k+1} (1-p)^{n-1-k} = np \sum_{k=0}^{n-1} (k+1) \frac{(n-1)!}{(n-1-k)!\,k!}\, p^k (1-p)^{n-1-k}$$

$= np\, \mathbb{E}(Y+1)$, con $Y \sim Bin(n-1, p)$

$= np\,(\mathbb{E}(Y)+1) = np[(n-1)p+1]$

$$\therefore Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = np[np-p+1] - (np)^2 = \cancel{(np)^2} + np(1-p) - \cancel{(np)^2}$$

$$\Rightarrow Var(X) = np(1-p)$$

P1) c)

- Notemos $X(X-1) = X^2 - X \Rightarrow \mathbb{E}(X^2) = \mathbb{E}[X(X-1)] + \mathbb{E}(X)$

↳ $\mathbb{E}(X(X-1)) = \sum_{k=1}^{\infty} k(k-1)\, p\, (1-p)^{k-1} = p \sum_{k=0}^{\infty} k(k-1)\, (1-p)^{k-1} = p(1-p) \sum_{k=0}^{\infty} k(k-1)(1-p)^{k-2}$

$= p(1-p) \sum_{k=0}^{\infty} + \frac{d^2}{dp^2}\left((1-p)^k\right) = p(1-p) \frac{d^2}{dp^2}\left[\sum_{k=0}^{\infty} (1-p)^k\right]$ * Serie geométrica

P1) c) $\mathbb{E}(X(X-1)) = p(1-p)\dfrac{d^2}{dp^2}\left[\sum_{k=0}^{\infty}(1-p)^k\right] = p(1-p)\dfrac{d^2}{dp^2}\left(\dfrac{1}{1-(1-p)}\right)$

$$= p(1-p)\frac{d^2}{dp^2}\left(\frac{1}{p}\right) = p(1-p)\frac{d}{dp}\left(-\frac{1}{p^2}\right) = p(1-p)\cdot 2\frac{1}{p^3}$$

$$= \frac{2(1-p)}{p^2}$$

$$\Rightarrow \mathbb{E}(X^2) = \mathbb{E}(X(X-1)) + \mathbb{E}(X) = \frac{2(1-p)}{p^2} + \frac{1}{p} = \frac{2(1-p)+p}{p^2}$$

$$= \frac{2-2p+p}{p^2} = \frac{2-p}{p^2}$$

$$\therefore \mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2} \quad \blacksquare$$

P1) d) $\mathbb{E}(X^2) = \sum_{k=0}^{\infty} k^2 \dfrac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda}\sum_{k=1}^{\infty} k^2\dfrac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=1}^{\infty} k\dfrac{\lambda^k}{(k-1)!}$

$$= e^{-\lambda}\sum_{k=0}^{\infty}(k+1)\frac{\lambda^{k+1}}{k!} = \lambda\sum_{k=0}^{\infty}(k+1)\frac{e^{-\lambda}\lambda^k}{k!}$$

$$= \lambda\,\mathbb{E}(X+1) = \lambda(\mathbb{E}(X)+1) = \lambda(\lambda+1) = \lambda^2+\lambda$$

$$\therefore \mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \lambda^2+\lambda-\lambda^2 = \lambda \quad \blacksquare$$

* Se puede hacer con (P3) del Aux 8

P1) e) $\mathbb{E}(X^2) = \int_a^b x^2 \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b x^2 dx = \frac{1}{b-a} \left[\frac{x^3}{3}\right]_a^b$

$= \frac{1}{b-a}\left(\frac{b^3}{3} - \frac{a^3}{3}\right) = \frac{b^3-a^3}{3(b-a)} = \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}} = \frac{b^2+ab+a^2}{3}$

$\therefore Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{a^2+ab+b^2}{3} - \frac{a^2+2ab+b^2}{4}$

$= \frac{4a^2+4ab+4b^2-3a^2-6ab-3b^2}{12} = \frac{a^2-2ab+b^2}{12} = \frac{(a-b)^2}{12}$ ∎

P1) f) Sea $Z \sim N(0,1) \Rightarrow Var(Z) = \mathbb{E}(Z^2) - \mathbb{E}(Z)^2 = \mathbb{E}(Z^2)$

$\Rightarrow \mathbb{E}(Z^2) = \int_{-\infty}^{+\infty} z^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = 2\int_0^{+\infty} z^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = 2\int_0^{+\infty} (-z)(-z) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$

$\left[\begin{array}{l} \text{IPP} \\ u = (-z) \longrightarrow du = -dz \\ dv = e^{-\frac{1}{2}z^2}\cdot(-z)dz \longrightarrow v = e^{-\frac{1}{2}z^2} \end{array}\right]$

$\Rightarrow \mathbb{E}(Z^2) = \frac{2}{\sqrt{2\pi}}\left[e^{-\frac{1}{2}z^2}\cdot(-z)\Big|_0^{+\infty} + \int_0^{+\infty} e^{-\frac{1}{2}z^2} dz\right]$

$= \frac{2}{\sqrt{2\pi}}\left(0 + \int_0^{+\infty} e^{-\frac{1}{2}z^2} dz\right) = 2\int_0^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = 2 \cdot \frac{1}{2} = 1$

$\therefore Var(Z) = 1$ //

Luego $X \sim N(\mu, \sigma^2) \Rightarrow X = \sigma Z + \mu$

$\therefore Var(X) = \sigma^2 Var(Z) = \sigma^2$ ∎

P1) g) $\mathbb{E}(X^2) = \int_0^{+\infty} x^2 \lambda e^{-\lambda x} dx = \int_0^{+\infty} -x^2 \cdot \frac{d}{dx}(e^{-\lambda x}) dx$

IPP

$u = x^2 \longrightarrow du = dx\, 2x$

$dv = \frac{d}{dx}(e^{-\lambda x}) dx \longrightarrow v = e^{-\lambda x}$

$$= -\left[x^2 e^{-\lambda x} \Big|_0^{+\infty} - \int_0^{+\infty} 2x e^{-\lambda x} dx \right] = \int_0^{+\infty} 2x e^{-\lambda x} dx$$

$$= \frac{2}{\lambda} \underbrace{\int_0^{+\infty} x \lambda e^{-\lambda x} dx}_{\mathbb{E}(X)} = \frac{2}{\lambda^2}$$

$\therefore \operatorname{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$ ∎

P1) h) $\mathbb{E}(X^2) = \int_0^{+\infty} x^2 \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx = \frac{\beta^{\alpha-1}}{\Gamma(\alpha)} \int_0^{+\infty} x^{\alpha+1} \cdot -\frac{d}{dx}(e^{-\beta x}) dx$

IPP

$u = x^{\alpha+1} \longrightarrow du = (\alpha+1) x^\alpha dx$

$dv = \frac{d}{dx}(e^{-\beta x}) dx \longrightarrow v = e^{-\beta x}$

$$= \frac{-\beta^{\alpha-1}}{\Gamma(\alpha)} \left[x^{\alpha+1} e^{-\beta x} \Big|_0^{+\infty} - \int_0^{+\infty} (\alpha+1) x^\alpha e^{-\beta x} dx \right]$$

$$= \frac{\beta^{\alpha-1}}{\Gamma(\alpha)} \int_0^{+\infty} (\alpha+1) x^\alpha e^{-\beta x} dx$$

$$= \frac{(\alpha+1)}{\beta} \underbrace{\int_0^{+\infty} x \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx}_{\mathbb{E}(X)} = \frac{(\alpha+1)\alpha}{\beta^2}$$

$\therefore \operatorname{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{(\alpha+1)\alpha}{\beta^2} - \frac{\alpha^2}{\beta^2} = \frac{\alpha^2 + \alpha - \alpha^2}{\beta^2} = \frac{\alpha}{\beta^2}$ ∎

P2) Si $Var(X) = 0 \Rightarrow \mathbb{E}((X - \mathbb{E}(X))^2) = 0$

• Como $[X - \mathbb{E}(X)]^2$ es una v.a. no negativa, por la (P2) del Aux 8:

$$\mathbb{E}[(X-\mathbb{E}(X))^2] = 0 \Rightarrow \mathbb{P}((X-\mathbb{E}(X))^2 = 0) = 1$$

$$\Rightarrow \mathbb{P}(X - \mathbb{E}(X) = 0) = 1$$

$$\Rightarrow \mathbb{P}(X = \mathbb{E}(X)) = 1$$

P3) Primero, como $X \sim Exp(\lambda)$ es una v.a. no negativa $\Rightarrow |X| = X$

Luego $\mathbb{E}(|X|^r) = \mathbb{E}(X^r) = \int_0^{+\infty} x^r \lambda e^{-\lambda x} dx = \int_0^{+\infty} x^r \cdot \frac{-d}{dx}(e^{-\lambda x}) dx$

IPP

$$\left.\begin{array}{l} u = x^r \longrightarrow du = r x^{r-1} dx \\ dv = \frac{d}{dx}(e^{-\lambda x}) dx \longrightarrow v = e^{-\lambda x} \end{array}\right\}$$

$$= -\left[x^r e^{-\lambda x} \Big|_0^{+\infty} - \int_0^{+\infty} r x^{r-1} e^{-\lambda x} dx \right]$$

$$= r \int_0^{+\infty} x^{r-1} e^{-\lambda x} dx = \frac{r}{\lambda} \int_0^{+\infty} x^{r-1} \lambda e^{-\lambda x} dx$$

$$= \frac{r}{\lambda} \mathbb{E}(X^{r-1})$$

$$\therefore \mathbb{E}(X^r) = \frac{r}{\lambda} \mathbb{E}(X^{r-1})$$

Luego por recursión, tenemos: $\mathbb{E}(X^r) = \frac{r!}{\lambda^r}$