Pauta T2

P1) a) PDQ) $\{V=n\} \iff \{\sum_{i=1}^{n-1} Y_i = r-1 , Y_n = 1\}$

$\Rightarrow$)

$$\{V=n\} \iff \inf_{k\geq 1}\{\sum_{i=1}^{k} Y_i = r\} = n$$

$$\Rightarrow \sum_{i=1}^{n} Y_i = r \wedge \forall k \in \{1 .. n-1\}: \sum_{i=1}^{k} Y_i < r$$

$$\Rightarrow \sum_{i=1}^{n-1} Y_i + Y_n = r \wedge \forall k \in \{1 .. n-1\}: \sum_{i=1}^{k} Y_i < r$$

$$\Rightarrow \sum_{i=1}^{n-1} Y_i < r , \sum_{i=1}^{n-1} Y_i + Y_n = r$$

$$\Rightarrow \sum_{i=1}^{n-1} Y_i = r-1 , Y_n = 1 \blacksquare$$

$\Leftarrow$)

$$\{\sum_{i=1}^{n-1} Y_i = r-1 , Y_n = 1\} \Rightarrow \sum_{i=1}^{n} Y_i = r , \sum_{i=1}^{n-1} Y_i = r-1$$

$$\Rightarrow \sum_{i=1}^{n} Y_i = r , \sum_{i=1}^{k} Y_i \leq r-1 \quad \forall k \in \{1 .. n-1\}$$

$$\Rightarrow \sum_{i=1}^{n} Y_i = r , \sum_{i=1}^{k} Y_i < r \quad \forall k \in \{1 .. n-1\}$$

$$\Rightarrow \inf_{k\geq 1}\{\sum_{i=1}^{k} Y_i = r\} = n$$

$$\Rightarrow \{V=n\} \blacksquare$$

P1) b) $\overset{n\geq r}{\mathbb{P}(V=n)} = \mathbb{P}(\sum_{i=1}^{n-1} Y_i = r-1 , Y_n = 1) \overset{\text{indep}}{=} \mathbb{P}(\sum_{i=1}^{n-1} Y_i = r-1)\, \mathbb{P}(Y_n = 1)$

$$= \binom{n-1}{r-1} p^{r-1} (1-p)^{n-1-(r-1)} \cdot p = \binom{n-1}{r-1} p^{r} (1-p)^{n-r} \blacksquare$$

P2) a) Sea $K \in \{-n+2j \mid j = 0 \ldots n\}$

$$\mathbb{P}(Y=K) = \mathbb{P}\left(\sum_{i=1}^{n} Y_i = K\right) = \mathbb{P}\left(\sum_{i=1}^{n}(2X_i - 1) = K\right) = \mathbb{P}\left(2\sum_{i=1}^{n} X_i - n = K\right)$$

$$= \mathbb{P}\left(\sum_{i=1}^{n} X_i = \frac{K+n}{2}\right) = \mathbb{P}\left(\sum_{i=1}^{n} X_i = j\right) \quad \text{con } j \in \{0 \ldots n\}$$

$$= \binom{n}{\frac{n+K}{2}} p^{\frac{K+n}{2}} (1-p)^{\frac{n-K}{2}} \quad \blacksquare$$

P2) b) Primero notemos que:

$$Y_i^1 + Y_i^2 = 0 \iff 2X_i^1 - 1 + 2X_i^2 - 1 = 0 \iff 2(X_i^1 + X_i^2) = 2$$

$$\iff X_i^1 + X_i^2 = 1$$

$$\therefore\ Z = \inf\{i \in \mathbb{N} : X_i^1 + X_i^2 = 1\}$$

~~Esto último es equivalente al desarrollo de la pregunta (P3 b),~~

Sea $n \in \mathbb{N}$

$$\Rightarrow \mathbb{P}(Z=n) = \mathbb{P}(\inf\{i \in \mathbb{N} \mid X_i^1 + X_i^2 = 1\} = n)$$

$$= \mathbb{P}(X_n^1 + X_n^2 = 1,\ X_i^1 + X_i^2 \neq 1\ \forall i \in \{1 \ldots n-1\})$$

$$\overset{\text{indep}}{=} \mathbb{P}(X_n^1 + X_n^2 = 1) \prod_{i=1}^{n-1} \mathbb{P}(X_i^1 + X_i^2 \neq 1)$$

$$\overset{\text{unión disj}}{=} \left[\mathbb{P}(X_n^1 = 0, X_n^2 = 1) + \mathbb{P}(X_n^1 = 1, X_n^2 = 0)\right] \prod_{i=1}^{n-1} \left[\mathbb{P}(X_i^1 = 0, X_i^2 = 0) + \mathbb{P}(X_i^1 = 1, X_i^2 = 1)\right]$$

$$\overset{\text{indep}}{=} \left[\mathbb{P}(X_n^1 = 0)\mathbb{P}(X_n^2 = 1) + \mathbb{P}(X_n^1 = 1)\mathbb{P}(X_n^2 = 0)\right] \prod_{i=1}^{n-1} \left[\mathbb{P}(X_i^1 = 0)\mathbb{P}(X_i^2 = 0) + \mathbb{P}(X_i^1 = 1)\mathbb{P}(X_i^2 = 1)\right]$$

$$= \left[(1-p)p + p(1-p)\right] \prod_{i=1}^{n-1} \left[(1-p)^2 + p^2\right]$$

$$= 2p(1-p)\left((1-p)^2 + p^2\right)^{n-1} \quad \blacksquare$$

P3) a) Primero, notemos que como $Y = \mathbb{1}_{U \leq p}$ es una indicatriz, los únicos valores posibles que puede tomar son 0 y 1:

- $\mathbb{P}(Y=1) = \mathbb{P}(\mathbb{1}_{U \leq p} = 1) = \mathbb{P}(U \leq p) = F_U(p) = \frac{p-0}{1-0} = p$ //
- $\mathbb{P}(Y=0) = \mathbb{P}(\mathbb{1}_{U \leq p} = 0) = \mathbb{P}(U > p) = 1 - \mathbb{P}(U \leq p) = 1 - p$ //

$\therefore Y \sim \text{Bernoulli}(p)$ ∎

P3) b) $Y = -\log(U)$

- Primero, notemos que $U \sim \text{unif}(0,1) \Rightarrow 0 < U(\omega) < 1 \quad \forall \omega \in \Omega$

$\Rightarrow \log(U) < 0 \quad \forall \omega \in \Omega \Rightarrow -\log(U) > 0 \quad \forall \omega \in \Omega$

$\therefore Y > 0 \quad \forall \omega \in \Omega$

- Sea $k > 0$:

$$\mathbb{P}(Y \leq k) = \mathbb{P}(-\log(U) \leq k) = \mathbb{P}(\log(U) \geq -k) = \mathbb{P}(U \geq e^{-k})$$

$$= 1 - \mathbb{P}(U < e^{-k}) \overset{\text{abs. cont}}{=} 1 - \mathbb{P}(U \leq e^{-k}) = 1 - e^{-k} = 1 - e^{-1 \cdot k}$$

$$\therefore F_Y(k) \begin{cases} = 1 - e^{-1 \cdot k} & \forall k > 0 \\ = 0 & \forall k \leq 0 \end{cases} \qquad \therefore Y \sim \text{Exp}(1)$$

∎

P4) a) Sea $K \in \mathbb{R}$

$\underline{K \geq 0)}$ Supongamos $X = K \Rightarrow \max\{X, 0\} = \max\{K, 0\} = K$

$$\Rightarrow \max\{-X, 0\} = \max\{-K, 0\} = 0$$

$$\Rightarrow X^+ - X^- = K - 0 = K \quad \therefore X = X^+ - X^-$$

$\underline{K < 0)}$ Supongo $X = K \Rightarrow X^+ = \max\{X, 0\} = \max\{K, 0\} = 0$

$$X^- = \max\{-X, 0\} = \max\{-K, 0\} = -K$$

$$\Rightarrow X^+ - X^- = 0 - (-K) = K \quad \therefore X = X^+ - X^-$$

$$\therefore X = X^+ - X^- \quad \forall K \in \mathbb{R}$$

P4) b) PDQ) F_{X^+} discont. en $0 \iff \mathbb{P}(X > 0) < 1$

Notemos que por def. $X^+ \geq 0$:

$\Rightarrow)$ Si F_{X^+} es discont. en $0 \iff \mathbb{P}(X^+ = 0) > 0$

Entonces $\mathbb{P}(X > 0) = \mathbb{P}(X^+ > 0) = 1 - \mathbb{P}(X^+ = 0) < 1$

$\Leftarrow)$ Con el mismo argumento

$\mathbb{P}(X > 0) < 1 \Rightarrow \mathbb{P}(X^+ > 0) < 1 \Rightarrow 1 - \mathbb{P}(X^+ = 0) < 1 \Rightarrow \mathbb{P}(X^+ = 0) > 0$

Luego F_{X^+} es discont. en 0

P4) c) Notemos, sea $K \in \mathbb{R}$:

$\cancel{F_{X^+}(K) = \mathbb{P}(X^+ \leq K) = 1 - \mathbb{P}(X^+ > K)}$

$\underline{K \geq 0)}$ $F_{X^+}(K) = \mathbb{P}(X^+ \leq K) = 1 - \mathbb{P}(X^+ > K) = 1 - \mathbb{P}(X > K)$

$$= \mathbb{P}(X \leq K) = F_X(K)$$

$\underline{K < 0)}$ $F_{X^+}(K) = \mathbb{P}(X^+ \leq K) = 0$, pues $X^+ \geq 0$

$$\therefore F_{X^+}(K) = \begin{cases} F_X(K) & K \geq 0 \\ 0 & K < 0 \end{cases}$$

Sea $k \in \mathbb{R}$

P5) a) $F_M(k) = \mathbb{P}(M \leq k) = 1 - \mathbb{P}(M > k) = 1 - \mathbb{P}(\min\{X_i \mid i = 1 \ldots n\} > k)$

$$= 1 - \mathbb{P}(X_i > k \ \forall i = 1 \ldots n)$$

$$\text{indep} = 1 - \prod_{i=1}^{n} \mathbb{P}(X_i > k)$$

$$= 1 - \prod_{i=1}^{n} (1 - \mathbb{P}(X_i \leq k))$$

$$= 1 - \prod_{i=1}^{n} \left[1 - (1 - e^{-\lambda k}) \mathbb{1}_{(0,\infty)}(k) \right] \Rightarrow \quad \bullet \text{ Si } k < 0: \ F_M(k) = 0$$

• Si $k > 0$: $F_M(k) = 1 - \prod_{i=1}^{n} \left[1 - (1 - e^{-\lambda k}) \right]$

$$= 1 - \prod_{i=1}^{n} e^{-\lambda k} = 1 - e^{-\lambda n k}$$

$$\therefore F_M(k) = \begin{cases} 1 - e^{-\lambda n k} & k \geq 0 \\ 0 & k < 0 \end{cases} \Rightarrow M \sim Exp(\lambda n)$$

P5) b) Como ya vimos en (a) que $M \sim Exp(\lambda n)$

$$\Rightarrow f_M(k) = \lambda n e^{-\lambda n k} \mathbb{1}_{(0,+\infty)}(k)$$

P5) c) Calculemos primero la distrib. de W; sea $k \in \mathbb{R}$:

$$F_W(k) = \mathbb{P}(W \leq k) = \mathbb{P}(X_1^{1/\beta} \leq k)$$

• Como $X_1 \sim \text{Exp}(\lambda) \Rightarrow \mathbb{P}(X_1^{1/\beta} \leq 0) = 0$, luego asumemos $k > 0$

$$\therefore F_W(k) = \mathbb{P}(X_1^{1/\beta} \leq k) = \mathbb{P}(X_1 \leq k^\beta) = 1 - e^{-\lambda k^\beta}$$

$$\Rightarrow F_W(k) = \begin{cases} 1 - e^{-\lambda k^\beta} & k \geq 0 \\ 0 & k < 0 \end{cases}$$

Luego $$f_W(k) = \frac{dF_W}{dk}(k) = \begin{cases} \lambda \beta k^{\beta-1} e^{-\lambda k^\beta} & k \geq 0 \\ 0 & k < 0 \end{cases}$$