Auxiliar 11.

P1

Si $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{l}\right) + b_n \cdot \sin\left(\frac{n\pi x}{l}\right)$, $\forall x \in [-l, l]$

Podemos obtener los coeficientes calculando

$$a_n = \frac{1}{l} \cdot \int_{-l}^{l} f(x) \cdot \cos\left(\frac{n\pi x}{l}\right) dx$$

$\cos(n\pi) = (-1)^n$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \cdot \sin\left(\frac{n\pi x}{l}\right) dx$$

Calculemos la serie de Fourier de $f(x) = x^2$ en $[-1,1]$. Como f par $b_n = 0$ $\forall n$.

$$a_0 = \int_{-1}^{1} x^2 dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$a_n = \int_{-1}^{1} x^2 \cdot \cos(n\pi x) \underset{\text{IPP}}{=} \underbrace{x^2 \cdot \frac{\sin(n\pi x)}{n\pi}\Big|_{-1}^{1}}_{0} - \int_{-1}^{1} 2x \cdot \frac{\sin(n\pi x)}{n\pi}$$

$$= \frac{-2}{n\pi}\left[-x\frac{\cos(n\pi x)}{n\pi}\Big|_{-1}^{1} + \int_{-1}^{1} \frac{\cos(n\pi x)}{n\pi} \right]$$

$\cancel{= \frac{2}{(n\pi)^2}\left[-\cos(n\pi) + \cos(n\pi) \right]}$

$$= \frac{-2}{(n\pi)^2}\left[-\cos(n\pi) - \cos(n\pi) + \underbrace{\frac{\sin(n\pi x)}{n\pi}\Big|_{-1}^{1}}_{0} \right]$$

$$= \frac{\cancel{2}\,4}{(n\pi)^2}\cdot\cos(n\pi) = \frac{4}{(n\pi)^2}\cdot(-1)^n.$$

$$\left\{\frac{(-1)^n}{n^2}\right\}$$

$$\Rightarrow f(x) = \frac{1}{3} + \frac{4}{(\pi)^2}\cdot\left(-\cos(\pi x) + \frac{1}{4}\cos(2\pi x) - \frac{1}{9}\cos(3\pi x) + \dots \right)$$

si tomamos $x=1$, como $f(x)=x^2 \Rightarrow f(1)=1$.

A su vez usando la serie de Fourier obtenemos

$$f(1) = \frac{1}{3} + \frac{4}{\pi^2}\cdot\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots\right)$$

$$\Rightarrow \left(\sum \frac{1}{n^2}\right)\cdot\frac{4}{\pi^2} + \frac{1}{3} = 1 \Rightarrow \left(\sum \frac{1}{n^2}\right)\frac{4}{\pi^2} = \frac{2}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

P2

a) Dada la función $f(x) = 1 - |x|$ calcule su serie de Fourier en $[-1,1]$.

$$a_0 = \int_{-1}^{1} 1 - |x| \, dx = 2 \int_0^1 1 - x \, dx = 2\left(x - \frac{x^2}{2} \Big|_0^1 \right)$$

$$= 2 \cdot \left(1 - \frac{1}{2}\right) = 1.$$

$$a_n = \int_{-1}^{1} (1 - |x|) \cdot \cos(n\pi x) = 2 \left[\int_0^1 \cos(n\pi x) - x \cdot \cos(n\pi x) \, dx \right]$$

$$= 2 \cdot \left[\underbrace{\frac{\sin(n\pi x)}{n\pi} \Big|_0^1}_{0} - \int_0^1 x \cdot \cos(n\pi x) \, dx \right] = -2 \cdot \left[\underbrace{x \cdot \frac{\sin(n\pi x)}{n\pi} \Big|_0^1}_{0} - \int_0^1 \frac{\sin(n\pi x)}{n\pi} \right]$$

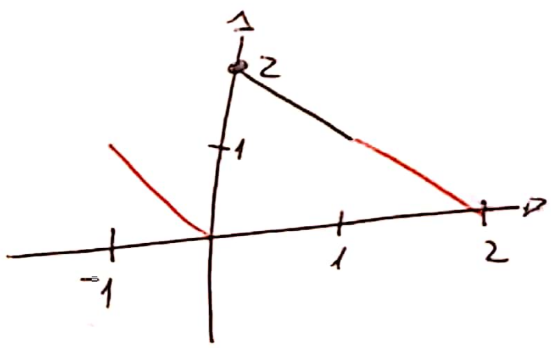
$$= +2 \left[+ \frac{-\cos(n\pi x)}{(n\pi)^2} \Big|_0^1 \right] = \frac{-2}{(n\pi)^2} \cdot \left(\cos(n\pi) - 1 \right)$$

$$= \frac{-2}{(n\pi)^2} \cdot \left((-1)^n - 1 \right)$$

$$= \frac{2}{(n\pi)^2} \left(1 - (-1)^n \right) = \begin{cases} 0 & n \text{ par} \\ \frac{4}{(n\pi)^2} & n \text{ impar} \end{cases}$$

$$\Rightarrow \quad f(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2 \pi^2} \cdot \cancel{\cos(n\pi x)} \cdot \cos((2k-1)\pi x).$$

b) $f(x) = 2-|x|$. Considerandola una función periodica de periodo 1:



$$f(x) = \begin{cases} 2-x & 1 > x \geq 0 \\ -x & -1 < x < 0 \end{cases}$$

$$a_0 = \int_{-1}^{1} f(x) = \int_{-1}^{0} -x\,dx + \int_{0}^{1} 2-x\,dx = \frac{-x^2}{2}\Big|_{-1}^{0} + 2x - \frac{x^2}{2}\Big|_{0}^{1}$$

$$= \frac{1}{2} + 2 - \frac{1}{2} = \underline{2}$$

4

$$a_n = \int_{-1}^{1} f(x) \cdot \cos(n\pi x)\, dx$$

$$= \int_{-1}^{0} -x \cdot \cos(n\pi x)\, dx + \int_{0}^{1} (2-x) \cos(n\pi x)\, dx$$

$$= \underbrace{\int_{-1}^{1} -x \cdot \cos(n\pi x)}_{0} + 2 \int_{0}^{1} \cos(n\pi x) = 2 \cdot \frac{\sin(n\pi x)}{\pi n}\Big|_{0}^{1} = 0.$$

$$b_n = \int_{-1}^{0} -x \cdot \sin(n\pi x)\, dx + \int_{0}^{1} (2-x) \cdot \sin(n\pi x)\, dx$$

$$= \int_{-1}^{1} -x \cdot \sin(n\pi x)\, dx + 2 \int_{0}^{1} \sin(n\pi x)\, dx$$

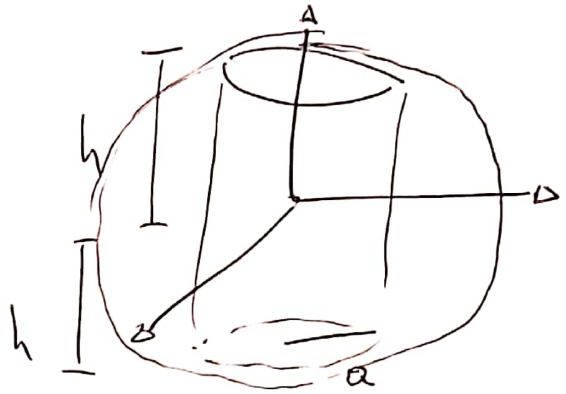
$$= \left(\frac{x \cdot \cos(n\pi x)}{n\pi}\Big|_{-1}^{1} + \underbrace{\int_{-1}^{1} \frac{\cos(n\pi x)}{n\pi}}_{0} \right) + 2 \cdot \frac{-\cos(n\pi x)}{n\pi}\Big|_{0}^{1}$$

$$= \frac{\cos(n\pi)}{n\pi} + \frac{\cos(n\pi)}{n\pi} + \frac{-2\cos(n\pi)}{n\pi} + 2 \cdot \frac{1}{n\pi}$$

$$= \frac{2}{n\pi}. \quad \Rightarrow f(x) = 1 + \sum_{n=1}^{\infty} \frac{2}{n\pi} \cdot \sin(n\pi x).$$

P3

$$E = \frac{Q}{4\pi\varepsilon_0} \cdot \frac{r}{r^2}$$



a) $R = \sqrt{a^2 + h^2}$

b) $\cos(\alpha) = h/R$

$$\phi \in [\alpha, \pi - \alpha]$$

$$\theta \in [0, 2\pi)$$

c) $\mathrm{div}(E) = \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(\frac{Q}{4\pi\varepsilon_0}\right)\right) = 0$

d)

$$\int_{\Omega} \mathrm{div}(E)\, dV = \int_{S_1} E \cdot \hat{n}\, dA + \int_{S_2} E\, \hat{n}\, dA = 0$$

$$\Rightarrow \int_{S_1} E \cdot \hat{e}\, dA = \int_{S_2} E \cdot \hat{r}\, dA$$

del cilindro

Calculamos el segundo

$$= \int_0^{2\pi} \int_{\alpha}^{\pi-\alpha} \frac{Q}{4\pi\varepsilon_0 r^2} \hat{r} \cdot \hat{r}\, r^2 \cdot \sin\phi\, d\phi\, d\theta$$

$$= \frac{Q}{\varepsilon_0} \cdot \frac{1}{\sqrt{\frac{a^2}{h^2}+1}} \cdot$$

Si $h \to \infty$

$$= \frac{Q}{\varepsilon_0}.$$

P4
XIII

$$F = 2y\hat{i} + 3x\hat{j} - z^2\hat{k}$$

$$rot(\vec{F}) = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

$$= (3 - 2)\hat{k} = \hat{k}.$$

Así,

$$\int_C \vec{F} \cdot \vec{dr} = \int_A \hat{k} \cdot \hat{r} \cdot dA.$$

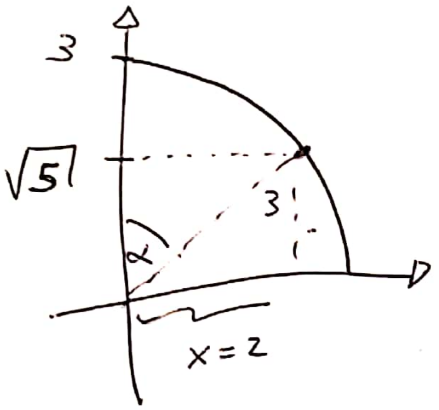
$$\hat{r} = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

$$\Rightarrow \hat{k} \cdot \hat{r} = \cos\varphi$$

$$= \int_A \cos\varphi \cdot dA$$

$$= \int_A \cos\varphi \cdot 3^2 \cdot \sin\varphi \cdot d\theta\, d\varphi \quad , \quad \theta \in [0, \pi/2]$$

¿Entre qué valores se mueve φ?



$x^2 + 5^2 = 3^2 \Rightarrow x = 2$

$\Rightarrow \alpha = \tan^{-1}\left(\frac{2}{\sqrt{5}}\right)$

La integral

$$= \int_0^{\pi/2} \int_0^{\alpha} \cos\varphi \cdot \sin\varphi \cdot 9 \, d\varphi \, d\theta$$

$$= \frac{9\pi}{2} \cdot \int_0^{\alpha} \cos(\varphi) \cdot \sin(\varphi) \, d\varphi$$

$$= \frac{9}{4}\pi \cdot \int_0^{\alpha} 2\cos\varphi \cdot \sin\varphi = \frac{9\pi}{4} \int_0^{\alpha} \sin(2\varphi) \, d\varphi$$

$$= \frac{9\pi}{8} \cdot -\cos(2\varphi) \Big|_0^{\alpha} = \frac{9\pi}{4} \cdot \left(\frac{1 - \cos 2\alpha}{2}\right)$$

$$= \frac{9\pi}{4} \cdot \sin^2\alpha = \frac{9\pi}{4} \cdot \underbrace{\left(\frac{2}{3}\right)^2}_{= \left(\frac{op}{H}\right)^2} = \frac{9\pi}{4} \cdot \frac{4}{9} = \boxed{\pi.}$$