P1 Calcular transformada de Fourier de

$$f(x) = \begin{cases} (x-3)e^{-4x} & , x \geq 3 \\ 0 & , x < 3 \end{cases}$$

Tenemos que

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix\xi} \, dx$$

$$= \overbrace{\frac{1}{\sqrt{2\pi}}}^{c} \int_3^{+\infty} (x-3) e^{-4x} e^{-ix\xi} \, dx = c \int_3^{+\infty} (x-3) e^{-x(4+i\xi)} \, dx$$

Por partes:

$$u = x-3 \qquad dv = e^{-x(4+i\xi)} dx$$

$$du = dx \qquad v = \frac{e^{-x(4+i\xi)}}{-(4+i\xi)}$$

Luego,

$$= c \left(\left. \frac{(x-3) e^{-x(4+i\xi)}}{-(4+i\xi)} \right|_3^{+\infty} + \int_3^{\infty} \frac{e^{-x(4+i\xi)}}{(4+i\xi)} dx \right)$$

$$= c \left(0 - \left. \frac{e^{-x(4+i\xi)}}{(4+i\xi)^2} \right|_3^{\infty} \right)$$

$$= -c \left(0 - \frac{e^{-3(4+i\xi)}}{(4+i\xi)^2} \right)$$

Concluimos que:

$$\boxed{\hat{f}(\xi) = \frac{e^{-3(4+i\xi)}}{\sqrt{2\pi}\,(4+i\xi)^2}. \qquad \xi \in \mathbb{R}}$$

P2 $f, g : \mathbb{R} \longrightarrow \mathbb{C}$ integrables t.q.

a) $\hat{f}$ y $\hat{g}$ también son integrables

El Teo de inversión nos dice que

$$f = \check{\hat{f}}$$

$$\int_{-\infty}^{\infty} \hat{f}(s)\,\overline{\hat{g}(s)}\,ds = \int_{-\infty}^{\infty} \hat{f}(s)\,\overline{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{-isx}\,dx}\,ds$$

$$* = \int_{-\infty}^{\infty} \hat{f}(s)\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{g(x)}\,e^{isx}\,dx\,ds$$

ordenamos con Fubini para obtener $\check{\hat{f}}$

$$= \int_{-\infty}^{\infty} \overline{g(x)}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(s)\,e^{isx}\,ds\right)dx$$

$$= \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\,dx.$$

b) Basta tomar $f = g$.

P3

$$f(x) = \begin{cases} 1 & , \ |x| < 1 \\ 0 & , \ \sim \end{cases}$$

Calculamos su transformada:

$$\hat{f}(s) = \overbrace{\frac{1}{\sqrt{2\pi}}}^{c} \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

$$= c \int_{-1}^{1} e^{-isx}\, dx = c \left. \frac{e^{-isx}}{-is} \right|_{-1}^{1}$$

$$= \frac{c}{-is} \left(e^{-is} - e^{is} \right) = \frac{c}{-is} \left(\cancel{\cos(s)} - i\,\mathrm{sen}(s) - [\cancel{\cos(s)} + i\,\mathrm{sen}(s)] \right)$$

$$= \frac{c\, 2\,\mathrm{sen}(s)}{s} = \frac{2\,\mathrm{sen}(s)}{\sqrt{2\pi}\, s}$$

Aplicamos la identidad anterior:

$$\int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |\hat{f}(s)|^2 ds$$

$$\int_{-1}^{1} 1 \, dx = \int_{-\infty}^{\infty} \frac{4 \, \operatorname{sen}^2(s)}{2\pi \, s} \, ds$$

$$\Rightarrow \quad \boxed{\int_{-\infty}^{\infty} \frac{\operatorname{sen}^2(s)}{s} \, ds = \pi}$$