P1

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ 0, & -\pi \leq x \leq 0 \end{cases}$$

$*$ f es L^2.

Pdq $$\sum_{n \text{ impar}} \frac{1}{n^2} = \frac{\pi^2}{8}$$

Lo primero que hacemos es calcular su serie de Fourier.

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(\xi)\,d\xi + \sum_{n=1}^{\infty}\left\{\Bigg[\underbrace{\frac{1}{l}\int_{-l}^{l} f(\xi)\cos\left(\frac{n\pi\xi}{l}\right)d\xi}_{a_n}\Bigg]\cos\left(\frac{n\pi x}{l}\right) + \Bigg[\underbrace{\frac{1}{l}\int_{-l}^{l} f(\xi)\,\mathrm{sen}\left(\frac{n\pi\xi}{l}\right)d\xi}_{b_n}\Bigg]\mathrm{sen}\left(\frac{n\pi x}{l}\right)\right\}$$

- Calculemos a_n:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\xi)\cos(n\xi)\,d\xi$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0\cos(n\xi)\,d\xi + \frac{1}{\pi}\int_{0}^{\pi} \underbrace{\xi\cos(n\xi)\,d\xi}_{x \text{ partes}}$$

$u = \xi$ $\quad dv = \cos(n\xi)\,d\xi$

$du = d\xi$ $\quad v = \frac{\mathrm{sen}(n\xi)}{n}$

$$= \frac{1}{\pi}\left(\xi \frac{\operatorname{sen}(n\xi)}{n} \Big|_0^{\pi} - \int_0^{\pi} \frac{\operatorname{sen}(n\xi)}{n} d\xi \right)$$

$$= \frac{-1}{\pi n} \frac{(-\cos(n\xi))}{n}\Big|_0^{\pi} = \frac{1}{\pi n^2} \cos(n\xi)\Big|_0^{\pi}$$

$$= \frac{1}{\pi n^2}\left((-1)^n - 1 \right)$$

- Calculamos a_0

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)\, d\xi = \frac{1}{2\pi}\int_0^{\pi} \xi\, d\xi = \frac{1}{2\pi} \frac{\pi^2}{2}$$

$$= \frac{\pi}{4}$$

- Calculamos b_n:

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(\xi) \operatorname{sen}\left(\frac{n\pi\xi}{\ell}\right) d\xi$$

x partes

$$= \frac{1}{\pi}\int_0^{\pi} \xi \operatorname{sen}(n\xi) d\xi$$

$u = \xi$ $\quad dv = \operatorname{sen}(n\xi)\, d\xi$

$du = d\xi$ $\quad v = \frac{-\cos(n\xi)}{n}$

$$= \frac{1}{\pi}\left(-\xi \frac{\cos(n\xi)}{n} \Big|_0^{\pi} + \int_0^{\pi} \frac{\cos(n\xi)}{n} d\xi \right)$$

$$= \frac{1}{\pi}\left(\frac{-\pi (-1)^n}{n} + \frac{\operatorname{sen}(n\xi)}{n^2}\Big|_0^{\pi} \right)$$

$$= \frac{1}{\pi}\left(-\frac{\pi(-1)^n}{n} + 0 \right) = -\frac{(-1)^n}{n}$$

Juntando obtenemos:

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{1}{\pi n^2} \underbrace{\left((-1)^n - 1 \right)}_{\substack{0,\ n \text{ par} \\ 2,\ n \text{ impar}}} \cos(nx)$$

$$+ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \operatorname{sen}(nx)$$

$$= \frac{\pi}{4} + \sum_{n \text{ impar}} \frac{1}{\pi n^2}\, 2 \underbrace{\cos(nx)}_{1 \text{ si } x=0}$$

$$+ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \underbrace{\operatorname{sen}(n x)}_{\text{muere si hacemos } x=0}$$

Tomamos $x = 0$:

$$f(0) \overset{x \text{ def}}{=} 0 = \frac{\pi}{4} - \sum_{n \text{ impar}} \frac{2}{\pi n^2}$$

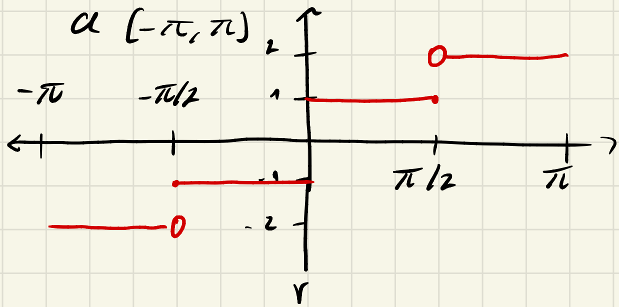
$$\Rightarrow \boxed{\sum_{n \text{ impar}} \frac{1}{n^2} = \frac{\pi}{8}}$$

P2

$$f(x) = \begin{cases} 1, & x \in [0, \pi/2] \\ 2, & x \in (\pi/2, \pi] \end{cases}$$

Extendemos para que sea impar:

a $[-\pi, \pi]$



$$\tilde{f}(x) = \begin{cases} f(x) & x \in [0, \pi] \\ -f(-x) & x \in [-\pi, 0) \end{cases}$$

* integral de función impar en dominio simétrico es 0

$$\int_{-\ell}^{\ell} g(x)\,dx = 0 \quad \text{si } g \text{ es impar.}$$

$$g(x) = -g(-x)$$

$$\Rightarrow a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\tilde{f}(x)\cos(nx)}_{\text{impar} \cdot \text{par} \,=\, \text{impar}}\,dx = 0$$

Para $n \in \{0, 1, 2, \dots\}$

Es decir, basta calcular los b_n.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \tilde{f}(x)\,\mathrm{sen}(nx)\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\,\mathrm{sen}(nx)\,dx$$

$$= \frac{2}{\pi}\left(\int_{0}^{\pi/2} \mathrm{sen}(nx)\,dx + \int_{\pi/2}^{\pi} 2\,\mathrm{sen}(nx)\,dx\right)$$

$$= \frac{2}{\pi}\left(-\frac{\cos(nx)}{n}\Big|_{0}^{\pi/2} - 2\,\frac{\cos(nx)}{n}\Big|_{\pi/2}^{\pi}\right)$$

$$= \frac{2}{\pi}\left(-\frac{\cos\left(\frac{n\pi}{2}\right)}{n} + \frac{1}{n} - \frac{2(-1)^n}{n} + \frac{2\cos\left(\frac{n\pi}{2}\right)}{n}\right)$$

$$= \frac{2}{\pi}\left(\frac{1}{n} - \frac{2(-1)^n}{n} + \frac{\cos\left(\frac{n\pi}{2}\right)}{n}\right)$$

Luego la serie de Fourier de f queda como

$$S_f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi n}\left(1 - 2(-1)^n + \cos\left(\frac{n\pi}{2}\right)\right)\mathrm{sen}(nx)$$

$$= \begin{cases} f(x), & x \in (0,\pi/2) \vee (\pi/2,\pi) \\ 3/2, & x = \pi/2 \\ 0, & x = \pi \vee x = 0 \end{cases}$$

P3 a) $f(x) = x^2$ en $[-1,1]$ $\ell = 1$

Calculamos $S_{f(x)}$. Como f es par,

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} \underbrace{\overset{\text{par}}{f(x)} \cdot \overset{\text{impar}}{\operatorname{sen}\left(\frac{n\pi x}{\ell}\right)}}_{\text{impar}} dx = 0$$

Luego sólo calculamos a_0 y a_n:

- $a_0 = 1\int_{-1}^{1} x^2 dx = 2\int_0^1 x^2 dx = \frac{2}{3}$.
- $a_n = 1\int_{-1}^{1} x^2 \cos(n\pi x) dx = 2\int_0^1 x^2 \cos(n\pi x) dx$

$= \underset{\text{dos veces}}{\text{x Partes}} = \frac{4}{(n\pi)^2}(-1)^n$

Concluimos:

$$S_f(x) = \frac{1}{\cancel{2}}\frac{\cancel{2}}{3} + \sum_{n=1}^{\infty}\frac{4}{(n\pi)^2}(-1)^n \cos(n\pi x).$$

b) $f(x) = e^x$ en $(0,\pi)$

Consideramos la extensión impar $\tilde{f}$

$$\tilde{f} = \begin{cases} f(x) & x \in [0,\pi] \\ -f(-x) & x \in [-\pi, 0] \end{cases}$$

Calculamos la serie $S_{\tilde{f}}$ con $\tilde{f}$ impar

$$\Rightarrow b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} \underbrace{\tilde{f}(x)\cos\left(\frac{n\pi x}{\ell}\right)}_{\text{impar}} dx = 0$$

$$a_0 = 1 \int_{-\pi}^{\pi} \tilde{f}(x)\, dx = 0$$

Luego basta calcular b_n:

$$b_n = 1 \int_{-\pi}^{\pi} \tilde{f}(x)\, \text{sen}(n x)\, dx$$

$$= \underbrace{2\int_0^{\pi} e^x \,\text{sen}(n\ x)\, dx}_{\text{por partes 2 veces}}$$

$$u = \text{sen}(n\ x) \qquad dv = e^x dx$$
$$du = \cos(nx)\, n\, dx \qquad v = e^x$$

$$\Rightarrow \int_0^{\pi} e^x \text{Sen}(n\ x)\, dx = \text{Sen}(n\ x) e^x \Big|_0^{\pi} - n\int_0^{\pi} e^x \cos(n\ x)\, dx$$

$$= -n\int_0^{\pi} e^x \cos(n\ x)\, dx \; ; \quad u = \cos(n\ x)$$
$$du = -\text{sen}(n\ x)\, n\, dx$$

$$dv = e^x dx$$
$$v = e^x$$

$$= -n\left(\cos(n\ x) e^x \Big|_0^{\pi} + \int_0^{\pi} n\, e^x \text{sen}(n\ x)\, dx\right)$$

$$= -n\left((-1)^n e^{\pi} - 1 + n\int_0^{\pi} e^x \operatorname{sen}(nx)\,dx \right)$$

$$= n\left(1 - (-1)^n e^{\pi}\right) - n^2 \int_0^{\pi} e^x \operatorname{sen}(nx)\,dx$$

$$\Rightarrow \int_0^{\pi} e^x \operatorname{sen}(nx)\,dx = \left(\frac{n}{1+n^2}\right)\left(1 - (-1)^n e^{\pi}\right)$$

Concluimos

$$S_f(x) = \sum_{n=1}^{\infty} \left(\frac{n}{1+n^2}\right) \frac{\left(1 - (-1)^n e^{\pi}\right)}{n} 2\operatorname{sen}(nx)$$