Auxiliar 11

P1

Optimizar $f(x,y)=x$ sobre $x^2+2y^2=3$

$x^2+2y^2-3=0$
$g(x,y)$

Lagrangeano $\mathcal{L}(x,y,\lambda)=f(x,y)-\lambda g(x,y)$

Cond optimalidad $\nabla\mathcal{L}(x,y,\lambda)=0=\begin{pmatrix}1-2\lambda x\\ 0-2\lambda y\\ x^2+2y^2-3\end{pmatrix}$ $\begin{matrix}(1)\\(2)\\(3)\end{matrix}$

$\overset{(2)}{\Rightarrow}$ $\lambda=0$ o $y=0$, si $\lambda=0$ $\ast$ $1_{\ast}$ (1)

$\Rightarrow$ $y=0$ y $\lambda\neq 0$ $\overset{(1)}{\Rightarrow}$ $\lambda=\frac{1}{2x}$

(3)

$\Rightarrow$ $x=\pm\sqrt{3}$ $\Rightarrow$ Hay 2 puntos

$\left(\sqrt{3},0,\frac{1}{2\sqrt{3}}\right)$

$\left(-\sqrt{3},0,-\frac{1}{2\sqrt{3}}\right)$

Veamos cual es máximo y cual mínimo ó silla

$H_{(x,y)}\mathcal{L}(x,y,\lambda)=\begin{bmatrix}-2\lambda & 0\\ 0 & -2\lambda\end{bmatrix}\Rightarrow H_{(x,y)}\mathcal{L}\left(\sqrt{3},0,\frac{1}{2\sqrt{3}}\right)=\begin{bmatrix}-\frac{1}{\sqrt{3}} & 0\\ 0 & -\frac{1}{\sqrt{3}}\end{bmatrix}$ def Negativa

$\Rightarrow$ es Máximo $(\sqrt{3},0)$

Similar $\overset{\text{def Positiva}}{\Longrightarrow}$ $(-\sqrt{3},0)$ es mínimo

2

La intersección de $2x - y + z = -\frac{1}{4}$ y $x^2+y^2=z$

$\Rightarrow -2x + y - \frac{1}{4} = z = x^2 + y^2$

$\Leftrightarrow x^2+2x+y^2-y+\frac{1}{4}=0$

$\Leftrightarrow x^2+2x+1+y^2-y+\frac{1}{4}-1=0$

$\Leftrightarrow (x+1)^2+(y-\frac{1}{2})^2 = 1$

$\Leftrightarrow g(x,y) = (x+1)^2+(y-\frac{1}{2})^2-1=0$

Ahora es un problema de 2 variables $\left(z = -\frac{1}{4} - 2x + y\right.$ también sirve $\left. z=x^2+y^2\right)$

$\Rightarrow f(x,y) = x+2y+3\left[-\frac{1}{4}-2x+y\right] = -\frac{3}{4} - 5x + 5y$

$\mathcal{L}(x,y,\lambda) = -\frac{3}{4} - 5x + 5y - \lambda\left((x+1)^2+(y-\frac{1}{2})^2-1\right)$

$\boxed{\nabla\mathcal{L} = \vec{0}}$

$$\begin{array}{c} (1) \\ (2) \\ (3) \end{array}\begin{pmatrix} -5 - \lambda\cdot 2\cdot(x+1) \\ 5 - \lambda\cdot 2\,(y-\frac{1}{2}) \\ (x+1)^2+(y-\frac{1}{2})^2-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Es fácil ver que $\lambda \neq 0$ pues $-5 \neq 0$

(1) $\Rightarrow (x+1) = \frac{-5}{2\lambda}$, (2) $\Rightarrow (y-\frac{1}{2}) = \frac{5}{2\lambda}$

(3) $\Rightarrow \frac{25}{4\lambda^2} + \frac{25}{4\lambda^2} - 1 = 0 \Rightarrow \frac{50}{4\lambda^2} = 1 \Rightarrow \frac{25}{2} = \lambda^2 \Rightarrow \boxed{\lambda = \pm\frac{5}{\sqrt{2}}}$

$\boxed{\lambda = \frac{5}{\sqrt{2}} \Rightarrow x = -\frac{\sqrt{2}}{2} - 1 \Rightarrow y = \frac{1}{2} + \frac{\sqrt{2}}{2} \Rightarrow z = \frac{9}{4} + \frac{3\sqrt{2}}{2}}$ (A)

$\boxed{\lambda = \frac{-5}{\sqrt{2}} \Rightarrow x = \frac{\sqrt{2}}{2} - 1 \Rightarrow y = \frac{1}{2} - \frac{\sqrt{2}}{2} \Rightarrow z = \frac{9}{4} - \frac{3}{2}\sqrt{2}}$ (B)

$$f(\star) = -\frac{3}{4} + 5\left(\frac{\sqrt{2}}{2}+1\right) + 5\left(\frac{\sqrt{2}}{2}+\frac{1}{2}\right)$$

$$= -\frac{3}{4} + 5 + \frac{5}{2} + 5\sqrt{2}$$

$$\boxed{f(\star) = \frac{27}{4} + 5\sqrt{2}}$$

$$f(\text{II}) = -\frac{3}{4} - \frac{5\sqrt{2}}{2} + 5 + \frac{5}{2} - \frac{5\sqrt{2}}{2}$$

$$= -\frac{3}{4} + \frac{15}{2} - 5\sqrt{2}$$

$$\boxed{f(\text{II}) = \frac{27}{4} - 5\sqrt{2}}$$

$f(\star)$ es el máximo

$$H_{x,y}\mathcal{L} = \begin{bmatrix} -2\lambda & 0 \\ 0 & -2\lambda \end{bmatrix}$$

$(\star) \to$ definido negativo — Máximo

$(\text{II}) \to$ definido positivo — Mínimo

P3

$$\mathcal{L}(\cancel{\ }x, \lambda) = x^t A x + c^t x - \lambda(a^t x - \beta)$$

$$\nabla_x \mathcal{L} = (A + A^t)x + c - \lambda a \overset{\text{imponemos}}{=} 0$$

$A + A^t = 2A$

$$\Rightarrow 2Ax^* + c - \lambda a = 0 \Rightarrow x^* = A^{-1}\frac{(\lambda a - c)}{2}$$

es def Positiva $\Rightarrow \det(A) > 0\ (\neq 0)$ por lo que A^{-1} existe

Por otro lado $a^t x^* = \beta \Rightarrow a^t \frac{\left[A^{-1}(\lambda a - c)\right]}{2} = \beta$

$$\Rightarrow \lambda\, a^t A^{-1} a - a^t A^{-1} c = 2\beta \Rightarrow \boxed{\lambda = \frac{2\beta + a^t A^{-1} c}{a^t A^{-1} a}}$$

$$\Rightarrow \boxed{x^* = \frac{A^{-1}}{2}\left[\frac{2\beta + a^t A^{-1} c}{a^t A^{-1} a} \cdot a - c\right]}$$

$H_{\mathcal{L}} = A + A^t = 2A$, Como A es definida positiva $\Rightarrow 2A$ lo es

$\Rightarrow x^*$ es mínimo

En particular para $a = (1, \dots, 1)$, $c = (0, \dots, 0)$, $A = I$, $\beta = 1$

$$x^* = \frac{I}{2}\left[\frac{2+0}{N} \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}\right] = \boxed{\begin{pmatrix} 1/N \\ 1/N \\ \vdots \\ 1/N \end{pmatrix}}$$

E4 Si $x_i = 0$ para algún $i \Rightarrow f(x_1, \dots, x_N) = 0$

Supongamos que $x_i \neq 0 \quad \forall i \in \{1, \dots, N\}$

$$\mathcal{L}(x_1, \dots, x_N, \lambda) = x_1 \cdot x_2 \cdot \dots \cdot x_N - \lambda \left(\sum_{k=1}^{N} \frac{x_k^2}{k^2} - 1 \right)$$

$$\nabla \mathcal{L} = \begin{pmatrix} \frac{x_1 \cdots x_N}{x_1} - \frac{2\lambda}{1^2} x_1 \\ \vdots \\ \frac{x_1 \cdots x_N}{x_i} - \frac{2\lambda}{i^2} x_i \\ \vdots \\ \frac{x_1 \cdots x_N}{x_N} - \frac{2\lambda x_N}{N^2} \end{pmatrix} \overset{\text{igualando}}{=} \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \quad x_1 \cdots x_N = 2\lambda \frac{x_i^2}{i^2} \quad \forall i \in \{1, \dots, N\}$$

$$\Rightarrow N \cdot x_1 \cdots x_N = \sum_{i=1}^{N} x_1 \cdots x_N = 2\lambda \sum_{i=1}^{N} \frac{x_i^2}{i^2} = 2\lambda \Rightarrow \boxed{x_1 \cdots x_N = \frac{2\lambda}{N}}$$

tomando i y j

$$\Rightarrow x_1 \cdots x_n = 2\lambda \frac{x_i^2}{i^2} \quad y \quad x_1 \cdots x_n = 2\lambda \frac{x_j^2}{j^2}$$

$$\Rightarrow 1 = \frac{x_i^2}{i^2} \cdot \frac{j^2}{x_j^2} \Rightarrow x_j^2 = \frac{x_i^2 \cdot j^2}{i^2} \Rightarrow \frac{x_j^2}{j^2} = \frac{x_i^2}{i^2} \quad (\smile)$$

$$\Rightarrow \cancel{x_j = \pm \left(x_i \cdot \frac{j}{i} \right)}$$

(☺) Sumando sobre i

$$\Rightarrow \underbrace{\sum_{i=1}^{n} \frac{x_j^2}{j^2}}_{\text{Constante}} = \sum_{i=1}^{n} \frac{x_i^2}{i^2} = 1 \Rightarrow \frac{x_j^2}{j^2} = \frac{1}{N} \Rightarrow x_j^2 = \frac{j^2}{N} \quad \forall j \in 1, \ldots, n$$

$$\Rightarrow (x_1 \cdots x_n)^2 = \frac{1}{N^N} \cdot (N!)^2 \Rightarrow x_1 \cdots x_n \nearrow \frac{N!}{N^{\frac{N}{2}}} \quad \text{Máximo}$$

$$\searrow -\frac{N!}{N^{\frac{N}{2}}} \quad \text{Mínimo}$$