Auxiliar 12

P1 a) $\displaystyle\int_0^1\int_0^1 xy\,e^{x+y}\,dy\,dx = \int_0^1 x e^x\left[\int_0^1 y e^y\,dy\right]dx$

$$= \int_0^1 x e^x\Big[e^y[y-1]\Big]_0^1\,dx \qquad e^1[1-1]-e^0[0-1]=1$$

$$= \int_0^1 x e^x \cdot 1\,dx = \boxed{1}$$

b)

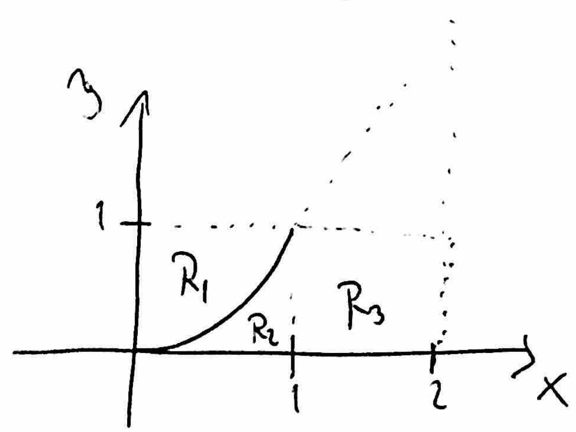
$$\int_0^1\int_0^2 \frac{8y}{x+1} - \frac{4x^3}{y^2+1}\,dx\,dy = \int_0^1\left(8y\left[\int_0^2 \frac{1}{x+1}\,dx\right] - \frac{1}{y^2+1}\left[\int_0^2 4x^3\,dx\right]\right)dy$$

$$= \int_0^1\left(8y\ln(3) - 2^4\cdot\frac{1}{y^2+1}\right)dy$$

$$= 4\ln(3) - 2^4\left[\operatorname{ArcTan}(1) - \operatorname{ArcTan}(0)\right] \qquad \operatorname{ArcTan}(1) = \frac{\pi}{4},\ \operatorname{ArcTan}(0) = 0$$

$$= 4\ln(3) - 2^4\cdot\left(\frac{\pi}{4}\right) = 4\left[\ln(3) - \pi\right]$$

c) $\displaystyle\int_0^2\int_0^1 \sqrt{|x^2-y|}\,dy\,dx =$ Complicado   Dibujemos



En R_1 $y > x^2$

En R_2 y R_3 $x^2 > y$

$[0,2]\times[0,1] = R_1 \cup R_2 \cup R_3$

Por lo tanto

$$\int_0^2\int_0^1 \sqrt{|x^2-y|}\,dy\,dx = \iint_{R_1}\sqrt{|x^2-y|} + \iint_{R_2}\sqrt{|x^2-y|} + \iint_{R_3}\sqrt{|x^2-y|}$$

$$= \iint_{R_1}\sqrt{y-x^2} + \iint_{R_2}\sqrt{x^2-y} + \iint_{R_3}\sqrt{x^2-y}$$

$\boxed{R_1}$

$x = \mathrm{Sin}(u)$
$dx = \mathrm{Cos}(u)\,du$

$$\iint_{R_1}\sqrt{y-x^2}\,dy\,dx = \int_0^1\int_{x^2}^1 \sqrt{y-x^2}\,dy\,dx = \int_0^1 (1-x^2)^{\frac{3}{2}}\cdot\frac{2}{3}\,dx = \frac{2}{3}\int_0^{\frac{\pi}{2}}\left(1-\mathrm{Sin}(u)^2\right)^{\frac{3}{2}}\cdot \mathrm{Cos}(u)\,du$$

Se usa $|\mathrm{Cos}(u)| = \mathrm{Cos}(u)$ para $u\in[0,\frac{\pi}{2}]$

$$= \frac{2}{3}\int_0^{\frac{\pi}{2}} \mathrm{Cos}(u)^4\,du = \frac{2}{3}\int_0^{\frac{\pi}{2}}\left(\frac{1+\mathrm{Cos}(2u)}{2}\right)^2 du = \frac{2}{3}\int_0^{\frac{\pi}{2}}\left(\frac{1}{4}+\frac{\mathrm{Cos}(2u)}{2}+\frac{\mathrm{Cos}^2(2u)}{4}\right)du$$

Propuesto

$$= \frac{2}{3}\int_0^{\frac{\pi}{2}}\left(\frac{1}{4}+\underbrace{\cancel{\frac{\mathrm{Cos}(2u)}{2}}}_{0}+\frac{1}{8}+\underbrace{\cancel{\frac{\mathrm{Cos}(4u)}{8}}}_{0}\right)du$$

$$= \frac{\cancel{2}}{\cancel{3}}\int_0^{\frac{\pi}{2}}\frac{\cancel{3}}{\cancel{8}}\,du = \frac{1}{4}\int_0^{\frac{\pi}{2}}du = \boxed{\frac{\pi}{8}}$$

$\boxed{R_2}$ $$\iint_{R_2} \sqrt{x^2-y}\,dy\,dx = \int_0^1\int_0^{x^2}\sqrt{x^2-y}\,dy\,dx = \int_0^1 \left[(x^2-y)^{\frac32}\left(-\frac23\right)\right]_0^{x^2} dx$$

$$= \int_0^1 x^3\left(\frac23\right)dx = \frac16 x^4\Big|_0^1 = \boxed{\frac16}$$

$\boxed{R_3}$ $$\iint_{R_3}\sqrt{x^2-y}\,dy\,dx = \int_1^2\int_0^1\sqrt{x^2-y}\,dy\,dx = \int_1^2\left[(x^2-y)^{\frac32}\left(-\frac23\right)\right]_0^1 dx$$

$$= \int_1^2 (x^2-1)^{\frac32}\left(-\frac23\right) + \left(\frac23\right)x^3\,dx$$

$$\boxed{\frac52}$$

$$= \left(-\frac23\right)\int_1^2 (x^2-1)^{\frac32}dx + \frac16(2^4-1)$$

$$\int_1^2 (x^2-1)^{\frac32}dx \overset{\substack{x=\cosh(u)\\ dx=\sinh(u)\,du}}{=}$$

También se puede hacer con el cambio $x = \sec u$, $dx = \sec(u)\tan(u)\,du$

Pero me da flojera

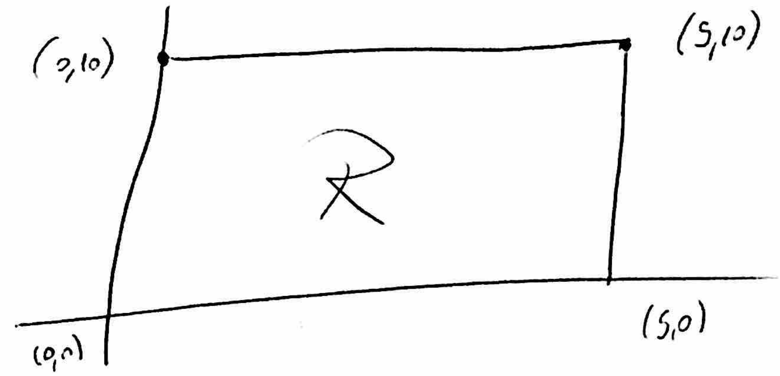
Debería ser $\frac38\left(2\sqrt3 + \cosh^{-1}(2)\right) \approx 1,7929$

$$\Longrightarrow \boxed{\int_0^2\int_0^1\sqrt{|x^2-y|}\,dy\,dx = \frac{\pi}{8} + \frac16 + \frac52 - \frac14\left(2\sqrt3 + \cosh^{-1}(2)\right)}$$

Nota: Quizás R_3 estaba más peludo, pero R_1 y R_2 deben aprender a calcularlos. ¿Por qué no se calculó directo?

P2

$$\iint_R x+y \, dy \, dx$$

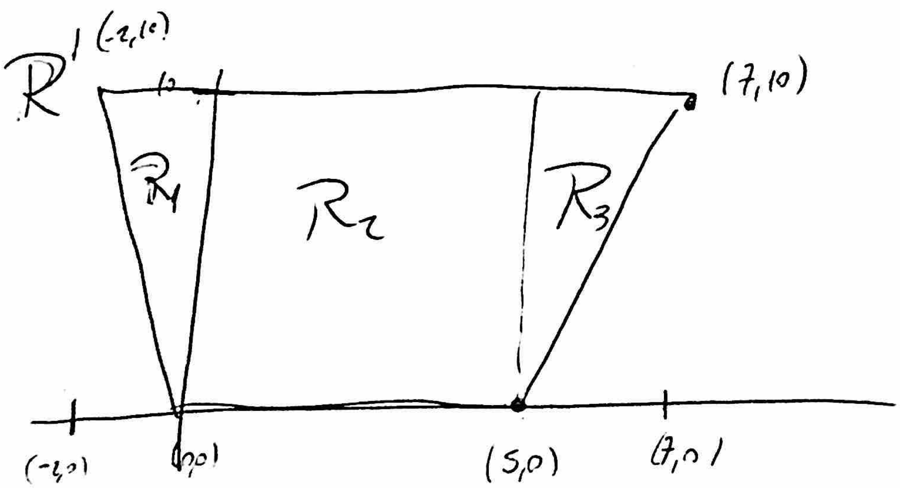


$$\boxed{R = \{(x,y)\in\mathbb{R}^2 : \quad 0 \le x \le 5 \,, \; 0 \le y \le 10\}}$$

$$\int_0^5 \int_0^{10} x+y \, dy \, dx = \int_0^5 x\left(\int_0^{10} dy\right)dx + \int_0^5 \left(\int_0^{10} y\,dy\right)dx$$

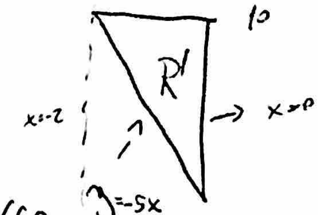
$$= \int_0^5 x \cdot 10 \, dx + \int_0^5 \frac{100}{2} \, dx$$

$$= 5^2 \cdot 5 + 50 \cdot 5 = 5^3(1+2) = 125 \cdot 3 = \boxed{375}$$



$$\iint_{R'} (x+y) \, dy \, dx = \iint_{R_1} (x+y) \, dy \, dx + \iint_{R_2} (x+y) \, dy \, dx + \iint_{R_3} (x+y) \, dy \, dx$$

$\boxed{R_1} = \{ (x,y) : \quad x \in [-2,0], \ y \in [-5x, 10] \}$



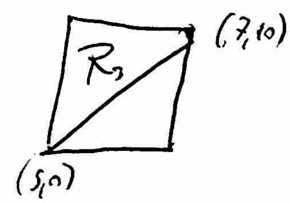
$$\int_{-2}^{0}\int_{-5x}^{10} x+y \, dy\, dx = \int_{-2}^{0} x \left(\int_{-5x}^{10} dy \right) dx + \int_{-2}^{0} \left(\int_{-5x}^{0} y\, dy \right) dx$$

$$= \int_{-2}^{0} x(10-5x)\, dx + \int_{-2}^{0} \frac{25x^2}{2} dx$$

$$= \int_{-2}^{0} 10x\, dx + \int_{-2}^{0} \frac{15x^2}{2} dx = 5x^2 + \frac{5x^3}{2} \Big|_{-2}^{0} = 0$$

$\boxed{R_2}$ Es la parte previa $\Rightarrow$ $\boxed{375}$

$\boxed{R_3} = \{ (x,y) : \quad x \in [5,7], \ y \in [5(x-5), 10] \}$



$$\int_{5}^{7}\int_{5(x-5)}^{10} x+y \, dy\, dx = \int_{5}^{7} x(10-5(x-5))\, dx + \int_{5}^{7} \frac{y^2}{2} \Big|_{5(x-5)}^{10} dx$$

$$= \cancel{\int_{5}^{7} 10x - 5x^2 + 25x \, dx + \int_{5}^{7} 50 + \left(\frac{-25}{2}\right)(x^2 - 10x + 25)\, dx}$$

$$\cancel{-\frac{25x^2}{2} + 125x + \frac{25^2}{2}}$$

$$\cancel{= \int_{5}^{7} \frac{25}{2}(-21)}$$

$\boxed{R_3}$

$$= \int_5^7 35x - 5x^2\,dx + \int_5^7 50 - \frac{25}{2}(x-5)^2\,dx$$

$$= \left[\frac{35}{2}x^2 - \frac{5}{3}x^3\right]_5^7 + 50\cdot 2 - \frac{25}{6}(x-5)^3\Big|_5^7$$

$$= \frac{35}{2}[7^2-5^2] - \frac{5}{3}[7^3-5^3] + 100 - \frac{25}{6}(2)^3$$

$$= \frac{35}{\cancel{2}}[\cancel{7-5}][7+5] - \frac{5}{3}[7-5][7^2+35+5^2] + 100 - \frac{100}{3}$$

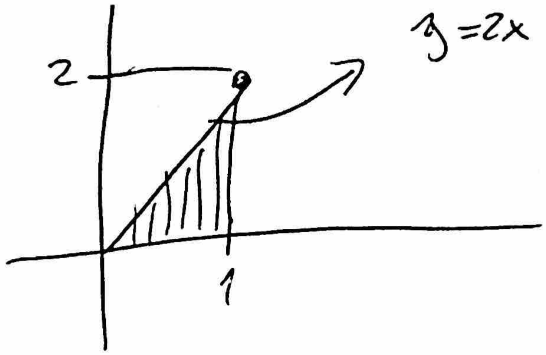
$$= 35\cdot 12 - \frac{10}{3}[109] + 100 - \frac{100}{3}$$

$$= 20[21+5] - \frac{10}{3}[109+10] = 20\cdot 26 - \frac{10}{3}[119] = \frac{10}{3}[156-119]$$

$$= \boxed{\frac{370}{3}}$$

$$\Rightarrow \iint_{R'} (x+y)\,dy\,dx = \boxed{0 + 375 + \frac{370}{3}}$$

$\boxed{P3}$ a) $\underbrace{\int_0^2 \int_{\frac{y}{2}}^1}_{R} y e^{x^3} \, dx \, dy$ imposible al tiro

$\Rightarrow R =$ 

$\Leftrightarrow x \in [0,1]$, $y \in [0,2x]$

$$\int_0^1 \int_0^{2x} y e^{x^3} \, dy \, dx = \int_0^1 e^{x^3} \left(\int_0^{2x} y \, dy \right) dx = \int_0^1 e^{x^3} \left[\frac{y^2}{2} \right]_0^{2x} dx$$

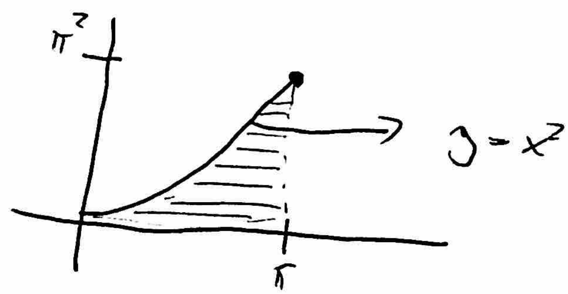
$$= \int_0^1 e^{x^3} (2x^2) \, dx$$

$u = x^3$

$du = 3x^2 dx$

$$= \frac{2}{3} \int_0^1 e^{x^3} (3x^2) \, dx = \frac{2}{3} e^{x^3} \Big|_0^1 = \boxed{\frac{2}{3}[e-1]}$$

b) $\int_0^{\pi^2} \int_{\sqrt{y}}^{\pi} y^{\frac{1}{2}} \frac{\sin(x^2)}{x^2} \, dx \, dy$

$\Omega =$ 

$\Leftrightarrow x \in [0,\pi]$, $y \in [0, x^2]$

$$\Rightarrow \int_0^{\pi} \int_0^{x^2} y^{\frac{1}{2}} \frac{\sin(x^2)}{x^2} \, dy \, dx = \int_0^{\pi} \frac{\sin(x^2)}{x^2} \cdot \left[y^{\frac{3}{2}} \cdot \left(\frac{2}{3}\right) \right]_0^{x^2} dx$$

$$= \int_0^{\pi} \frac{\sin(x^2)}{\cancel{x^2}} x^{\cancel{3}} \left(\frac{2}{3}\right) dx = \frac{1}{3} \int_0^{\pi} \sin(x^2)(2x) \, dx = \frac{1}{3} \sin(x^2) \Big|_0^{\pi}$$

$$= \boxed{\frac{\sin(\pi^2)}{3}}$$