P4

$$\frac{\partial x}{\partial \theta} = -\rho \sin\theta \quad , \quad \frac{\partial y}{\partial \theta} = \rho\cos\theta \quad , \quad \frac{\partial x}{\partial \rho} = \cos\theta \quad , \quad \frac{\partial y}{\partial \rho} = \sin\theta$$

$$\frac{\partial g}{\partial \rho} = \frac{\partial f}{\partial \rho} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \rho} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \rho}$$

$$\boxed{\frac{\partial g}{\partial \rho} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\cdot\sin\theta}$$

$\frac{\partial g}{\partial \rho}$ derivando nuevamente por $\frac{\partial}{\partial \rho}$

$$\frac{\partial^2 g}{\partial \rho^2} = \frac{\partial}{\partial \rho}\left(\frac{\partial f}{\partial x}\cdot\cos\theta + \frac{\partial f}{\partial y}\cdot\sin\theta\right) \qquad / \theta \text{ es cts para } \rho$$

$$= \frac{\partial}{\partial \rho}\left(\frac{\partial f}{\partial x}\right)\cdot\cos\theta + \frac{\partial}{\partial \rho}\left(\frac{\partial f}{\partial y}\right)\sin\theta$$

$$= \frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial x}{\partial \rho}\cdot\cos\theta + \frac{\partial^2 f}{\partial x\partial y}\cdot\frac{\partial y}{\partial \rho}\cdot\cos\theta + \frac{\partial^2 f}{\partial x\partial y}\cdot\frac{\partial x}{\partial \rho}\cdot\sin\theta + \frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial y}{\partial \rho}\cdot\sin\theta$$

$$\boxed{\frac{\partial^2 g}{\partial \rho^2} = \frac{\partial^2 f}{\partial x^2}(\cos\theta)^2 + 2\frac{\partial^2 f}{\partial x\partial y}\cdot\cos\theta\sin\theta + \frac{\partial^2 f}{\partial y^2}(\sin\theta)^2}$$

$$\frac{\partial g}{\partial \theta} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial f}{\partial x}(-\rho\sin\theta) + \frac{\partial f}{\partial y}(\rho\cos\theta)$$

$$\frac{\partial^2 g}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left[\frac{\partial f}{\partial x}(-\rho\cos\theta) + \frac{\partial f}{\partial y}(\rho\sin\theta)\right] = \rho\left[\frac{\partial^2 f}{\partial \theta\partial x}(-\cos\theta) + \frac{\partial f}{\partial x}\sin\theta + \frac{\partial^2 f}{\partial \theta\partial y}\sin\theta + \frac{\partial f}{\partial y}\cos\theta\right]$$

$$= \rho\left[(-\cos\theta)\left(\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial x\partial y}\cdot\frac{\partial y}{\partial \theta}\right) + \frac{\partial f}{\partial x}\sin\theta + (\sin\theta)\left(\frac{\partial^2 f}{\partial x\partial y}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial y}{\partial \theta}\right) + \frac{\partial f}{\partial y}\cos\theta\right]$$

$$= \rho\left[(-\cos\theta)\left(\frac{\partial^2 f}{\partial x^2}(-\rho\sin\theta) + \frac{\partial^2 f}{\partial x\partial y}\cdot(\rho\cos\theta)\right) + \frac{\partial f}{\partial x}\sin\theta + (\sin\theta)\left(\frac{\partial^2 f}{\partial x\partial y}(-\rho\sin\theta) + \frac{\partial^2 f}{\partial y^2}\cdot(\rho\cos\theta)\right) + \frac{\partial f}{\partial y}\cos\theta\right]$$

$$\frac{\partial^2 g}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left[\frac{\partial f}{\partial x}(-\rho \sin\theta) + \frac{\partial f}{\partial y}(\rho \cos\theta)\right] = \rho\left[\frac{\partial^2 f}{\partial \theta \partial x}(-\sin\theta) + \frac{\partial f}{\partial x}(-\cos\theta) + \frac{\partial^2 f}{\partial \theta \partial y}\cos\theta + \frac{\partial f}{\partial y}(-\sin\theta)\right]$$

$$= \rho\left[\left(\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial x \partial y}\cdot\frac{\partial y}{\partial \theta}\right)(-\sin\theta) + \frac{\partial f}{\partial x}(-\cos\theta) + \left(\frac{\partial^2 f}{\partial x \partial y}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial y}{\partial \theta}\right)(\cos\theta) + \frac{\partial f}{\partial y}(-\sin\theta)\right]$$

$$= \rho\left[\left(\frac{\partial^2 f}{\partial x^2}(-\rho\sin\theta) + \frac{\partial^2 f}{\partial x \partial y}(\rho\cos\theta)\right)(-\sin\theta) + \frac{\partial f}{\partial x}(-\cos\theta) + \left(\frac{\partial^2 f}{\partial x \partial y}\cdot(-\rho\sin\theta) + \frac{\partial^2 f}{\partial y^2}(\rho\cos\theta)\right)(\cos\theta) + \frac{\partial f}{\partial y}(-\sin\theta)\right]$$

$$\boxed{\frac{\partial^2 g}{\partial \theta^2} = \frac{\partial^2 f}{\partial x^2}\cdot\rho^2(\sin\theta)^2 - 2\rho^2\cos\theta\sin\theta\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}\rho^2(\cos\theta)^2 - \frac{\partial f}{\partial x}\rho\cos\theta - \frac{\partial f}{\partial y}\rho\sin\theta}$$

$$\Rightarrow \frac{\partial^2 g}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial g}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 g}{\partial \theta^2} = \frac{\partial^2 f}{\partial x^2}\overset{1}{\cancel{(\cos^2\theta + \sin^2\theta)}} + \frac{\partial^2 f}{\partial x \partial y}\overset{0}{\cancel{\left(2\cos\theta\sin\theta - \frac{2\rho^2}{\rho^2}\cos\theta\sin\theta\right)}} + \frac{\partial f}{\partial x}\overset{0}{\cancel{\left(\frac{1}{\rho} - \frac{\rho}{\rho^2}\right)}} + \frac{\partial f}{\partial y}\overset{0}{\cancel{\left(\frac{1}{\rho} - \frac{\rho}{\rho^2}\right)}}$$

$$+ \frac{\partial^2 f}{\partial y^2}\overset{1}{\cancel{(\cos^2\theta + \sin^2\theta)}}$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \Delta f$$, con lo que se concluye