$\boxed{P1}$

Pauta Auxiliar 7

$$\frac{\partial x}{\partial u} = u \; , \; \frac{\partial y}{\partial u} = v \; , \; \frac{\partial x}{\partial v} = -v \; , \; \frac{\partial y}{\partial v} = u$$

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial f}{\partial x}\cdot u + \frac{\partial f}{\partial y}\cdot v$$

$$\frac{\partial^2 g}{\partial^2 u} = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\cdot u + \frac{\partial f}{\partial y}\cdot v\right) = \frac{\partial^2 f}{\partial u \partial x}\cdot u + \frac{\partial f}{\partial x}\cdot\frac{\partial u}{\partial u} + \frac{\partial^2 f}{\partial u \partial y}\cdot v$$

$$= \left[\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y \partial x}\cdot\frac{\partial y}{\partial u}\right]u + \frac{\partial f}{\partial x} + \left[\frac{\partial^2 f}{\partial x \partial y}\cdot\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial y}{\partial u}\right]v$$

$$= \frac{\partial^2 f}{\partial x^2}u^2 + 2\frac{\partial^2 f}{\partial y \partial x}\cdot uv + \frac{\partial^2 f}{\partial y^2}\cdot v^2 + \frac{\partial f}{\partial x}$$

Se usa que f es C^2: $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

$$\frac{\partial g}{\partial v} = \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial f}{\partial x}(-v) + \frac{\partial f}{\partial y}u$$

$$\frac{\partial^2 g}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial x}(-v) + \frac{\partial f}{\partial y}\cdot u\right) = \frac{\partial^2 f}{\partial v \partial x}\cdot(-v) + \frac{\partial f}{\partial x}\frac{\partial(-v)}{\partial v} + \frac{\partial^2 f}{\partial v \partial y}\cdot u$$

$$= \left[\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial v}{\partial x} + \frac{\partial^2 f}{\partial y \partial x}\cdot\frac{\partial y}{\partial x}\right](-v) - \frac{\partial f}{\partial x} + \left[\frac{\partial^2 f}{\partial x \partial y}\cdot\frac{\partial x}{\partial v} + \frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial y}{\partial v}\right]u$$

$$= \frac{\partial^2 f}{\partial x^2}(-v)^2 - 2uv\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}u^2 - \frac{\partial f}{\partial x}$$

$$\Rightarrow \frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} = \left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right](u^2+v^2)$$

$$\Rightarrow \frac{1}{(u^2+v^2)}\left[\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2}\right] = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

P2

$u = f(x) + f(y)^2$

$v = f(x) + f(y)^2 + f(z)^3$

$$\frac{\partial h}{\partial x} = \frac{\partial g}{\partial x}(u,v) = \frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial g}{\partial u}(u,v)\cdot f'(x) + \frac{\partial g}{\partial v}(u,v)\cdot f'(x)$$

$$\frac{\partial h}{\partial x}(0,0,0) = \frac{\partial g}{\partial u}\left(f(0)+f(0)^2, f(0)+f(0)^2+f(0)^3\right)\cdot f'(0) + \frac{\partial g}{\partial v}\left(f(0)+f(0)^2, f(0)+f(0)^2+f(0)^3\right)\cdot f'(0)$$

$$= \frac{\partial g}{\partial u}(0,0)\cdot 1 + \frac{\partial g}{\partial v}(0,0)\cdot 1$$

/ $\nabla g(0,0) = \left(\frac{\partial g}{\partial u}(0,0), \frac{\partial g}{\partial v}(0,0)\right) = (1,3)$

$$= 1+3 = \boxed{4}$$

$$\frac{\partial h}{\partial y} = \frac{\partial g}{\partial y}(u,v) = \frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= \frac{\partial g}{\partial u}\cdot 2\cdot f(y)\cdot f'(y) + \frac{\partial g}{\partial v}\cdot 2\cdot f(y)\cdot f'(y)$$

/ Usando que $f(0)=0$

$$\frac{\partial h(0,0,0)}{\partial y} = \frac{\partial g}{\partial u}\cdot 0 + \frac{\partial g}{\partial v}\cdot 0 = 0$$

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial z}(u,v) = \frac{\partial g}{\partial u}\cdot \cancel{\frac{\partial u}{\partial z}}^{0} + \frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial z} = \frac{\partial g}{\partial v}\cdot 3f(z)^2\cdot f'(z)$$

$$\frac{\partial h}{\partial z}(0,0,0) = \frac{\partial g}{\partial v}\cdot 3f(0)^2\cdot f'(0) = 0$$

$$\Rightarrow \nabla h(0,0,0) = (4,0,0)$$

$\boxed{P3}$

$$g(\lambda) = f(\lambda x) = \lambda^p f(x)$$

$$\Rightarrow \frac{\partial g(\lambda)}{\partial \lambda} = g'(\lambda) = \frac{\partial}{\partial \lambda} f(\lambda x) = \frac{\partial}{\partial \lambda} f(\lambda x_1, \lambda x_2, \dots, \lambda x_N) = \frac{\partial}{\partial \lambda} f(u_1, u_2, \dots, u_N)$$

$$= \frac{\partial}{\partial u_1} f(u_1, \dots, u_N) \cdot \frac{\partial u_1}{\partial \lambda} + \dots \dots \frac{\partial}{\partial u_N} \cdot f(u_1, \dots, u_N) \cdot \frac{\partial u_N}{\partial \lambda}$$

$$= \sum_{i=1}^{N} \frac{\partial}{\partial u_i} f(u_1, \dots, u_N) \cdot \frac{\partial u_i}{\partial \lambda} = \sum_{i=1}^{N} \frac{\partial}{\partial u_i} f(u_1, \dots, u_N) \cdot x_i \quad , \text{pues } \boxed{u_i = \lambda x_i}$$

$$\Rightarrow g'(1) \overset{\text{evaluo } \boxed{\lambda=1} \Rightarrow u_i = 1 \cdot x_i = x_i}{=} \sum_{i=1}^{N} \frac{\partial}{\partial x_i} f(x_1, \dots, x_N) \cdot x_i = x \cdot \nabla f(x) \qquad (1)$$

Pero

$$g(\lambda) = \lambda^p f(x)$$

$$\Rightarrow g'(\lambda) = p \cdot \lambda^{p-1} \cdot f(x)$$

$$\Rightarrow g'(1) = p \cdot f(x) \qquad (2)$$

(1) con (2)

$$\Rightarrow \boxed{x \cdot \nabla f(x) = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(x_1, \dots, x_N) \cdot x_i = p f(x)}$$

P4) Primero notar que $\frac{\partial f}{\partial x}(x+y) = \frac{\partial f(u)}{\partial x} = \frac{\partial f(u)}{\partial u} \cdot \frac{\partial u}{\partial x} = f'(u) \cdot 1$

$$= f'(u) = f'(x+y)$$

$$\Rightarrow \frac{\partial f}{\partial x}(x+y) = f'(x+y)$$

Similar

$$\Rightarrow \frac{\partial f}{\partial y}(x+y) = f'(x+y)$$

$$\frac{\partial f}{\partial x}(x+y) = \frac{\partial f}{\partial y}(x+y)$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(xf(x+y) + y\, g(x+y)\right) = \frac{\partial x}{\partial x} \cdot f(x+y) + x \cdot \frac{\partial}{\partial x} f(x+y) + y \frac{\partial g}{\partial x}(x+y)$$

$$= f(x+y) + x\, f'(x+y) + y\, g'(x+y)$$ / Usando lo previo

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(f(x+y) + x\, f'(x+y) + y\, g'(x+y)\right)$$

$$= f'(x+y) + \cancel{\frac{\partial x}{\partial x}}^{1} f'(x+y) + x \cdot \frac{\partial}{\partial x} f'(x+y) + y \frac{\partial g'}{\partial x}(x+y)$$

$$= \boxed{2f'(x+y) + x\, f''(x+y) + y\, g''(x+y)}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(f(x+y) + x\, f'(x+y) + y\, g'(x+y)\right)$$

$$= \boxed{f'(x+y) + x\, f''(x+y) + g'(x+y) + y\, g''(x+y)}$$

$$\frac{\partial z}{\partial y} = x\, f'(x+y) + g(x+y) + y\, g'(x+y)$$

$$\frac{\partial^2 z}{\partial y} = \boxed{x\, f''(x+y) + 2g'(x+y) + y\, g''(x+y)}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$ (Reemplazar y Verificar ☺)

Propuesta 1

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y}(f(x,y)^2, y) = \frac{\partial f}{\partial y}(u,v) = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= \frac{\partial f(u,v)}{\partial u}\cdot\frac{\partial (f(x,y))^2}{\partial y} + \frac{\partial f(u,v)}{\partial v}\cdot\cancel{\frac{\partial y}{\partial y}}^{1}$$

$$= \frac{\partial f(u,v)}{\partial u}\cdot 2f(x,y)\cdot\frac{\partial f(x,y)}{\partial y} + \frac{\partial f(u,v)}{\partial v}$$

$$\frac{\partial F(1,2)}{\partial y} \Rightarrow \begin{matrix} x=1 \\ y=2 \end{matrix} \Rightarrow f(1,2)=1 \Rightarrow (f(1,2))^2 = 1 \Rightarrow f((f(1,2))^2, 2) = f(1,2) = 1$$

$$\nabla f(1,2) = \left(\frac{\partial f}{\partial u}(1,2), \frac{\partial f}{\partial v}(1,2)\right) = (4,2)$$

$$\Rightarrow \frac{\partial F(1,2)}{\partial y} = 4\cdot 2\cdot 1\cdot 2 + 2 = \boxed{18}$$

Propuesta 2

$$\frac{\partial f}{\partial x}(x,y,z) = (\sin(z), 0), \quad \frac{\partial f}{\partial z} = (x\cos(z), -y\sin(z))$$

$$\frac{\partial f}{\partial y}(x,y,z) = (0, \cos(z)), \quad \frac{\partial g}{\partial x} = -y^2\sin(x)$$

$$\frac{\partial g}{\partial y} = 2y\cos(x), \quad \frac{\partial h}{\partial x} = -y\,\frac{1}{\cos(x)}\cdot(-\sin(x))$$

$$\frac{\partial h}{\partial y} = -\ln(\cos(x))$$

$$\varphi_1(x,y) = f_1(x+y, g(x+y), h(y,x))$$

$$\varphi_2(x,y) = f_2(x+y, g(x+y, h(y,x))$$

$$\Rightarrow \frac{\partial \varphi_1}{\partial x} = \frac{\partial f_1}{\partial x}(x+y,\ g(x,y), h(y,x))$$

$$= \frac{\partial f_1}{\partial x}(u,v,w) = \frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f_1}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial f_1}{\partial w}\cdot\frac{\partial w}{\partial x}$$

$h(y,x) = -x\ln(\cos(y))$

$$= \frac{\partial f_1}{\partial u}(u,v,w)\cdot 1 + \frac{\partial f_1}{\partial v}(u,v,w)\cdot\frac{\partial g}{\partial x}(x,y) + \frac{\partial f_1}{\partial w}(u,v,w)\cdot\frac{\partial h(y,x)}{\partial x}$$

Ojo esto al reves

$$\frac{\partial \varphi_1}{\partial x}(x,y) = \sin(w) + 0 + u\cos(w)\cdot(-\ln(\cos(y)))$$

$$= \sin(h(y,x)) + (x+y)\cos(h(y,x))(-\ln(\cos(y)))$$

$$= \boxed{\sin(-x\ln(\cos(y))) - (x+y)\cos(-x\ln(\cos(y)))\ln(\cos(y))}$$

$$\frac{\partial \varphi_1}{\partial x}(0,1) = \sin(0) - (1)\cos(0)\ln(\cos(1)) = \boxed{-\ln(\cos(1))}$$

$$\frac{\partial \varphi_1}{\partial y} = \frac{\partial f_1}{\partial u}(u,v,w)\cdot\frac{\partial u}{\partial y} + \frac{\partial f_1}{\partial v}(u,v,w)\cdot\frac{\partial v}{\partial y} + \frac{\partial f_1(u,v,w)}{\partial w}\cdot\frac{\partial w}{\partial y}$$

$$= \sin(w)\cdot 1 + 0 + u\cos(w)\cdot\frac{\partial h(y,x)}{\partial y}$$

$$= \boxed{\sin(-x\ln(\cos(y))) + \frac{x\sin(y)}{\cos(y)}}$$

$$\frac{\partial \varphi_1}{\partial y}(0,1) = \sin(0) + 0\cdot\tan(1) = \boxed{0}$$

$$\frac{\partial \varphi_2}{\partial x} = \frac{\partial f_2}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f_2}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f_2}{\partial w} \cdot \frac{\partial w}{\partial x}$$

$$= 0 + \cos(w) \cdot \frac{\partial g(x,y)}{\partial x} - y \sin(w) \cdot \frac{\partial h(y,x)}{\partial x}$$

$$= \cos(w) \cdot (-y^2 \sin(x)) + y \sin(w) \ln(\cos(y))$$

$$= \cancel{-y^2 \sin(x) \; y \cos(w)}$$

$$= \boxed{-y^2 \cos(w) \sin(x) + y \sin(w) \ln(\cos(y))}$$

$$\frac{\partial \varphi_2}{\partial x}(0,1) = 0 + 1 \cdot \sin(h(1,0)) \ln(\cos(1))$$

$$= 0 + \sin(0) \cdot \ln(\cos(1)) = \boxed{0}$$

$$\frac{\partial \varphi_2}{\partial y} = \frac{\partial f_2}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f_2}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial f_2}{\partial w} \cdot \frac{\partial w}{\partial y}$$

$$= 0 + \cos(w)\, 2y \cos(x) + (-y \sin(w)) \frac{\partial}{\partial y}(-x \ln(\cos(y)))$$

$$= \cos(-x \ln(\cos(y)))\, 2y \cos(x) - y \sin(-x \ln(\cos(y))) \frac{x}{\cos(y)} \sin(y)$$

$$\frac{\partial \varphi_2}{\partial y}(0,1) = \cos(0) \cdot 2 \cdot \cos(0) - 1 \cdot \cancel{\sin(0) \cdot 0 \cdot \tan(1)}^{0} = \boxed{2}$$

$$J_\varphi(0,1) = \begin{bmatrix} -\ln(\cos(1)) & 0 \\ 0 & 2 \end{bmatrix}$$